Uglješa Bugarić
Assistant

Josif Vuković
Professor

University of Belgrade
Faculty of Mechanical Engineering

Optimal Control of Motion of the System Based on Mathematical Pendulum with Constant Length

The paper discusses the motion of the mathematical pendulum (with reference to the transportation machines modelling), from the state of constant velocity motion of the suspension point to the state of rest for the pre-assigned time with damping of oscillations at the end of the process. Solutions of the task were found by application of the Pontryagin's principle of maximum, and adaptive and digital methods of oscillations damping. Acceleration of the suspension point of the mathematical pendulum is used as the control value in all cases. The case of the constant length of the mathematical pendulum is discussed

Keywords: *Optimal control, oscillation damping, bulk cargo unloading.*

1. INTRODUCTION

Modelling of operation of the crane's systems (unloading bridges) can be done by establishing the analogy between a crane trolley with a grab on one side, and a mathematical pendulum with a moving point of suspension, on the other side. In this case the point of suspension of the mathematical pendulum represents the centre of gravity of the crane trolley, while the material point represents the grab with the payload.

Works of some authors like Kazak [12], [13], Komarov [14], Lobov [16] confirms this analogy, where the crane trolley and the load were modelled as dynamic systems with two masses inter-linked with an inelastic rope. In these works the laws of load oscillations in vertical planes are determined, as well as the recurrent influence of given oscillations on inertial forces of the system at various rules of change of the driving force of the crane trolley.

The task of reducing the influence of inertial force, by applying the above mentioned analogy, requires solving of two aspects of the motion of the mathematical pendulum: (1) motion of the mathematical pendulum from the state of rest to the assigned distance with damping of oscillations at the end of the process, and (2) motion of the mathematical pendulum from the state of rest at the beginning of acceleration period or state of motion at constant velocity at the beginning of braking period, to the state of uniform motion at an assigned velocity at the end of the acceleration period or to the state of rest at the end of the braking period, with damping of oscillations at the end of both periods of non-stationary motion modes (acceleration, braking).

Zaremba [29] solves the first task concerning the motion of mathematical pendulum of constant length for the minimal time, where velocity and acceleration (control values) are limited with absolute value. Chiong [6] takes into consideration velocity of the suspension point limited with absolute value and obtains the solution for the "first" and the "second" aspect of the pendulum motion through the viscous environment for a minimum time, which is a control value. Bolotnik and Chiong [3], solve the "first" aspect of the motion of the mathematical pendulum of variable length taking for the control value the driving force limited with an absolute value.

The second approach to the aspect of motion of mathematical pendulum is by using the Pontryagin's principle of maximum. Taking the limited acceleration of the suspension point as the control value, and applying the principle of maximum, Sokolov [27] defines that the choice of the value of control in the given moment, determines whether the value of functional (which presents the energy of mathematical pendulum) would be minimal or maximal. Karihaloo and Parbery [10], solve the first issue of the motion of the mathematical pendulum by taking the driving force as the control value, and applying the principle of maximum. The functional, which should be minimised in this case within the given interval of time, is the control itself - the square of driving force. Zrnić et al. [30], solve the "second" issue of the motion of the mathematical pendulum by using the principle of maximum, taking the acceleration of the suspension point as the control value. The functional, which should be minimised within the given interval of time, represents the total of squares of inclination angle, angular velocity of inclination and control (acceleration).

A considerably different approach to the solutions of issues of dynamic loads of crane systems, damping of oscillations during the motion of crane trolley and load can be found in the papers of: Auering [1], [2], Schwardtmann [23], Unbehauen et al. [28]. Authors of these papers model the control systems on the basis of known characteristics of crane systems and their dynamic models, and determine regimes of work of driving mechanisms with the aim of oscillations damping, i.e. automation of crane facilities. That practically means that the automation of crane facilities

Received: October 2001, accepted: March 2002.
Correspondence to:
Uglješa Bugarić, Faculty of Mechanical Engineering,
27. marta 80, 11000 Belgrade, Yugoslavia

is possible only if it is possible to control load oscillations.

Moustafa and Ebeid [17] derive a non-linear model for an overhead crane, which takes into account simultaneous travel and transverse motions of the crane. They also develop an anti-swing control system, which adopts a feedback control to specify the crane speed at every moment.

One of the ways of damping of oscillations in mechanical systems is shown in the papers: Hyde and Seering [8], Singer and Seering [24], [25], [26] where the general theory on damping of oscillations of mechanical (flexible) systems, based on assignment of a corresponding, desired input to the system ("input shaping") is developed. This desired input is applied not only in the coupling with the open loop systems, but also in the coupling with closed loop systems. This approach is developed in the Massachusetts Institute of Technology (MIT).

In the papers of Noakes et al. [19], [20], Noakes and Jansen [18] and Kress et al. [15] application of the general theory of damping of oscillations, developed in MIT, to the damping of oscillations of the load suspended by the rope and transported by the overhead crane is presented. This load represents a pendulum which can freely oscillate during transportation and whose oscillations are damped. This way of damping of oscillations is applied and realised in Oak Ridge National Laboratory (ORNL).

It is also important to mention the research done in Sandia National Laboratories (SNL), which shows that it is possible to damp oscillations of a load suspended on a rope if the time of constant acceleration / deceleration is equal to the period of natural oscillations of the load, i.e. if the acceleration / deceleration of crane is programmed in an appropriate manner. These algorithms represent, in fact, open loop systems. The results of researches and applications can be found in the work of Jones and Petterson [9].

The above mentioned papers are based on the following facts (Carbon, [5]): basically, there are three characteristical types of device distinguished by taking the acceleration / deceleration time as a reference variable: 1. the natural oscillation period of the load (pendulum) is smaller than the acceleration / deceleration time, 2. the natural oscillation period of pendulum is larger than the acceleration / deceleration time, 3. the natural oscillation period of pendulum is equal to the acceleration / deceleration time. The following damping methods can be employed on the basis of these values (see figure 2.): a) digital oscillation damping, b) analog oscillation damping, and c) adaptive oscillation damping.

In this paper the "second" issue of the motion of a mathematical pendulum is solved for the period of deceleration (braking) of the mathematical pendulum point of suspension (crane trolley). Optimal solutions are researched by using the principle of maximum for a system with constant length of the mathematical pendulum. At the same time, solutions, which could be obtained by using adaptive and digital methods of oscillations damping, are also shown. They are given in analytical form.

2. A FEW WORDS ABOUT UNLOADING OF THE BULK CARGO WITH GRAB CRANE DEVICES

Bulk cargo terminal represents the organisation of different activities, connected with handling and control of material flow from the vessel to the transport or storage system, with maximum servicing of vessels at minimum expenses. The feature of the bulk cargo is the fact that the costs of transportation, manipulation and waiting represent an important part of their values. The terminal operates 24 hours a day, seven days a week during the sailing period. Even the very small reduction of the duration of the unloading cycle can save energy needed for the unloading cycle and increase unloading capacity. [30]

Unloading devices are in most cases, bottlenecks of the terminal, so their optimal function is the basic prerequisite for the optimum performance of the whole system. Bulk cargo can be unloaded by continuos unloading devices, or by grab crane devices.

This paper considers only the unloading cycle of grab crane devices. Automation of the unloading process of the crane facilities with grab is possible, but very expensive. On the other side, the crane operator can not repeat the optimal unloading cycle in the longer time period. The only practical and feasible solution is to introduce the semi-automatic unloading cycle which consists of the manual part, where the crane operator controls the grab motion, and the automatic part where the computer controls the grab moving according to the given algorithm [21].

The manual part of the semi-automatic unloading cycle consists of lowering of the empty grab to the material surface in the vessel, from the point of completion of the automatic part of the unloading cycle, garbing of the material and hoisting of the grab with cargo to the point of commencement of the automatic part of the unloading cycle. The automatic part of the semi-automatic unloading cycle consists of the grab transfer from the point of commencement of the automatic part of the unloading cycle to the receiving hopper, grab discharging and empty grab return from the hopper to the point of completion of the automatic part of the unloading cycle. The commencement / completion point of the automatic part of the semi-automatic unloading cycle depends on given geometry of the system, river water level, material level in the vessel, etc. In this paper only the automatic part of the semi-automatic unloading cycle will be analysed.

3. MATHEMATICAL MODEL

Planary motion of the given dynamic system, idealising a crane plant installation, shown on figure 1. will be considered. A rigid body of mass m_1 is connected by massless, inextensible rope with another rigid body of mass m_2 performing a rectilinear motion under the driving force F_d. The direction of motion of mass m_2 is chosen as the horizontal "z"-axis. It is assumed that the driving force F_d can be directed along both the positive and the negative "z"-direction. That could be achieved in practice by using an "ac" motor. The suspended mass m_1 performs motion in the "zy"-plane under the tension in the rope and the gravitational force.

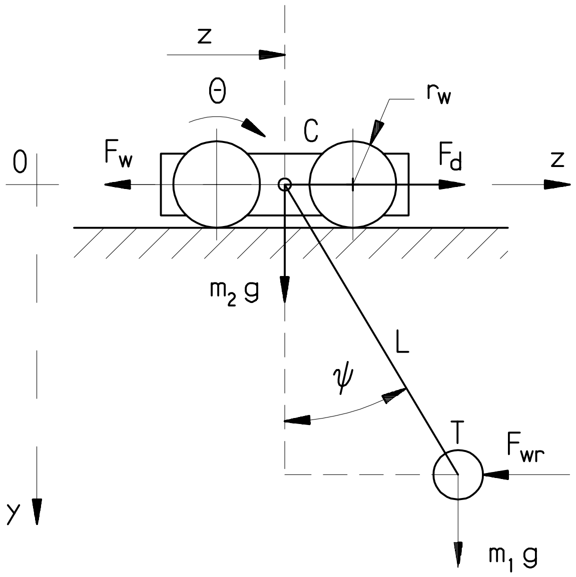

Figure 1. Simplified crane's trolley and cargo moving scheme.

Review of indications used in the mathematical model: z - instantaneous centre of gravity position of the crane trolley, ψ - angle of vertical rope inclination, g - gravity acceleration, $m_1 = 12500\,\text{kg}$ - mass of the cargo and grab, $m_w = 253.5\,\text{kg}$ - mass of the trolley wheel, m_2=15000 kg - mass of the crane trolley, L - instantaneous length of rope (length of rope is changing during the time of grab hoisting and lowering), $r_w = 0.2\,\text{m}$ - radius of the trolley wheel, θ - angle of wheel twisting ($z=\theta\cdot r_w$), F_d - driving force of the trolley, F_{wr} - resistance to wind, acting on the grab, depends on the grab surface (neglected), F_w - resistance to trolley motion:

$$F_w = g\,(m_1 + m_2)\cdot(2\,f + \mu\,d)\cdot\beta / D_w;\ [31] \qquad (1)$$

where d=0.1 m - the trolley wheel axis diameter, $D_w=2\cdot r_w$ - the trolley wheel diameter, μ=0.012 - coefficient of friction between the wheel bearing and the axle journal, $f=0.05\cdot10^{-2}$ m - coefficient of rolling friction, β=2.3 - flange friction factor (caused by trolley skewing).

Using the Lagrange's equations we obtain the system of two differential equations describing the motion of the grab and the crane trolley. For small angles of rope inclination $\psi\in(0\div10)°$ the system can be written as: [4], [30]

- first equation:

$$\begin{aligned}&(m_1 + 2\cdot m_w + m_2)\cdot\ddot{z} + m_1\cdot L\cdot\ddot{\psi} + m_1\cdot\psi\cdot\ddot{L}\\ &- m_1\cdot L\cdot\psi\cdot\dot{\psi}^2 + 2\cdot m_1\cdot\dot{L}\cdot\dot{\psi} = F_d - F_w - F_{wr}\end{aligned} \qquad (2)$$

-second equation:

$$\ddot{z} + L\cdot\ddot{\psi} + 2\cdot\dot{L}\cdot\dot{\psi} = -g\cdot\psi - F_{wr}/m_1;$$

The second equation of the obtained system is the differential equation of the mathematical pendulum of variable length, with moving point of suspension with the material point affected by force of resistance to motion.

4. CONTROL OF TROLLEY AND GRAB MOTION DURING OSCILLATIONS DAMPING

Taking into account theoretical considerations of adaptive and digital damping of oscillations, the second equation of system the (2) will be considered separately.

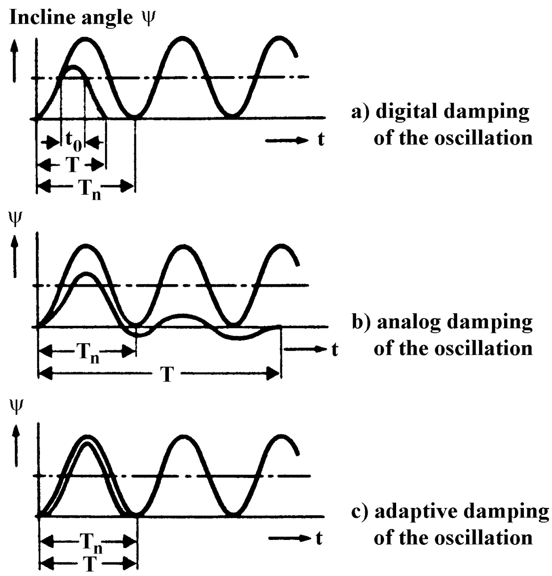

Figure 2. Damping methods of pendulum oscillation. T_n - period of natural oscillations of pendulum, T - period of acceleration or braking, t_0 - period during which acceleration / barking is interrupted.

If we adopt that $\ddot{z}(t) = const.$, which means that the velocity of the point of suspension of mathematical pendulum (centre of gravity of the crane trolley) is changing with constant acceleration / deceleration. For achieving the adaptive damping of oscillations (figure 2c), it is necessary to define, by solving the second differential equation of the system (2), the duration of the acceleration / braking period at the end of which the angle of inclination and angular velocity of rope inclination are equal to zero.

Digital damping of oscillations can be realised through three different stages of motion of the crane trolley centre of gravity, (figure 2a). If we adopt that $\ddot{z}(t) = const.$ during the first stage in the second equation of the system (2), the velocity of the point of suspension is changing with constant acceleration / deceleration. There is no acceleration $\ddot{z}(t) = const.$ during the second stage of motion (period t_0, figure 2), so the velocity of suspension point is equal to the velocity attained at the end of the first stage. If we adopt that in the third stage $\ddot{z}(t) = const.$, the velocity of suspension point is changing with constant acceleration / deceleration from the initial velocity, equal to the one at the end of the second stage, to some required final velocity of motion at the end of acceleration period or to the state of rest at the end of braking period. It is necessary to define the duration of each particular stage by solving the system of transcendent equations obtained by solving the second differential equation of the system (2) for each stage with corresponding boundary conditions. The condition, which must be used for solving the given system, is, as in adaptive damping, that the angle of rope inclination and angular velocity of rope inclination should equal zero at the end of the acceleration / braking period.

The constant obtained from the condition that $\ddot{z}(t) = const.$, which means that the acceleration / deceleration of suspension point is constant, i.e. that the velocity of point of suspension is changing with uniform acceleration / deceleration, is equal to the relation between the required velocity at the end / start of the period (given stage) of acceleration / braking and the duration of acceleration / braking period (given stage). The constant $\ddot{z}(t)$ can be also calculated as the tangent of angle between the straight line of the change of velocity of the point of suspension and the time axis.

For obtaining the optimal oscillation damping in the non-stationary modes of movement (acceleration, braking), the second equation of the system (2) should be also separately observed and can be written as: (for $L=const.$, $F_{wr}=0$)

$$\ddot{\psi} = -(g/L)\cdot\psi - \ddot{z}/L\,; \tag{3}$$

Taking the crane trolley acceleration $\ddot{z}$ for the control and introducing the new variables:

$$\psi = y_1; \dot{\psi} = y_2; \dot{z} = y_3; \ddot{z} = u, \tag{4}$$

the equation (3) can be replaced by the system of equations of the first order:

$$\dot{y}_1 = y_2\,;\ \dot{y}_2 = -(g/L)\cdot y_1 - u/L\,;\ \dot{y}_3 = u\,; \tag{5}$$

The problem consists of defining such a manner of control "u" for which the system, for the acceleration period, from the initial state:

$$t = t_0,\ y_1(t_0) = 0,\ y_2(t_0) = 0,\ y_3(t_0) = 0\,; \tag{6a}$$

will arrive to the state:

$$t = t_c,\ y_1(t_c) = 0,\ y_2(t_c) = 0,\ y_c(t_c) = V_t\,, \tag{6b}$$

or, for the braking period, from initial state:

$$t = t_0, y_1(t_0) = 0,\ y_2(t_0) = 0, y_3(t_0) = V_t\,; \tag{7a}$$

to the state:

$$t = t_c,\ y_1(t_c) = 0,\ y_2(t_c) = 0,\ y_3(t_c) = 0\,; \tag{7b}$$

where V_t - crane trolley velocity in the stationary mode. The time interval $[t_0, t_c]$ is known in advance.

The unique control is conditioned by the requirement:

$$\int_{t_0}^{t_c} \frac{1}{2}(y_1^2 + y_2^2 + u^2)\,\mathrm{d}t \to \min. \tag{8}$$

This condition of optimality prevents the values of control and rope inclination angle from becoming too high. The problem defined by relations (5), (6), (7) and (8) is reduced to the form which enables direct application of the principle of maximum [22]. To that end and in compliance with (5) and (8) the function

$$\begin{aligned} \mathrm{H} = \lambda_1 y_2 + \lambda_2 \left[-(g/L)\cdot y_1 - u/L\right] + \\ + \lambda_3 u - (y_1^2 + y_2^2 + u^2)/2; \end{aligned} \tag{9}$$

is established where the values λ_1, λ_2, λ_3 satisfy the differential equations system:

$$\dot{\lambda}_1 = -\partial H/\partial y_1, \dot{\lambda}_2 = -\partial H/\partial y_2, \dot{\lambda}_3 = -\partial H/\partial y_3\,. \tag{10}$$

According to the theorem of the principle of maximum, the function (9) has the maximum value as the optimal solution. According to the required condition of extreme:

$$\partial H/\partial u = 0\ \ ;\ \ \partial^2 H/\partial u^2 > 0\ , \tag{11}$$

the control

$$u = \ddot{z} = -(1/L)\cdot\lambda_2 + \lambda_3\ , \tag{12}$$

is obtained.

By substituting (8) in the equations (5) and (9), the following equation system is obtained:

$$\begin{aligned} &\dot{y}_1 = y_2\,; \\ &\dot{y}_2 = -(g/L)\,y_1 + (1/L^2)\,\lambda_2 - (1/L)\,\lambda_3\,; \\ &\dot{y}_3 = -(1/L)\,\lambda_2 + \lambda_3\ ;\ \dot{\lambda}_1 = (g/L)\,\lambda_2 + y_1\,; \\ &\dot{\lambda}_2 = -\lambda_1 + y_2\ ;\ \dot{\lambda}_3 = 0\,; \end{aligned} \tag{13}$$

and for its solution there is the sufficient number of conditions (6) and (7). The system of differential equations defined in this way, with conditions (6) and (7), represents a two-point boundary value problem.

Issue of the motion of the mathematical pendulum (crane trolley and load) will be solved for boundary conditions defined by expression (7), for the breaking period, and for the case of constant length of the mathematical pendulum (rope), because there is no difference in the form of solutions if we solve the said issue with boundary conditions (6) or (7).

The change of driving force of the suspension point of the mathematical pendulum (crane trolley) which is necessary for realisation of a required motion in time, should be obtained by replacing the corresponding values for rope inclination angle (ψ), angular velocity ($\dot{\psi}$) and angular acceleration of rope inclination ($\ddot{\psi}$), velocity ($\dot{z}$) and acceleration (control) ($\ddot{z}$ i.e. u) of the suspension point, which are obtained in one of the three shown ways, in the first equation of the differential equations system (2).

4.1. Damping of Oscillations of the Mathematical Pendulum with Constant Rope Length

Differential equation (3), for the change of the velocity of the point of suspension with constant acceleration / deceleration ($\ddot{z}(t) = const.$), is transformed to the following form:

$$L\,\ddot{\psi} + g\,\psi = -c \;; \tag{14}$$

The solution of the differential equation (14) has the form: [11]

$$\psi = C_1 \cos(k\,t) + C_2 \sin(k\,t) - c\,/\,g\;; \tag{15}$$

After the differentiation of the expression (15), it follows that the angular velocity of rope inclination

$$\dot{\psi} = -C_1 k \sin(k\,t) + C_2 k \cos(k\,t)\;; \tag{16}$$

where C_1, C_2 - constants which should be determined from the system of equations (15) and (16) on the base of initial conditions, and $k = \sqrt{g\,/\,L}$ - angular frequency of mathematical pendulum oscillations.

a) *Adaptive damping of oscillations of the mathematical pendulum with constant rope length* [4]

At the adaptive damping of oscillations with constant length of the mathematical pendulum (rope) and boundary conditions defined by expressions (7) (braking), the constant "*c*" ($\ddot{z}(t) = const.$) has the following value:

$$c = V_t\,/\,T\;;$$

where V_t - velocity of the point of suspension at the start of the braking period, and T - braking time.

Initial conditions, on the basis of which the constants C_1 and C_2 are calculated, are:

$$\psi(0)=0 \;\text{ and }\; \dot{\psi}(0)=0\;;$$

which means that at the initial moment of the braking period the mathematical pendulum is in the equilibrium position.

By substituting the expression for constant "*c*" and initial conditions in the expressions (15) and (16) the values for constants C_1 and C_2 are obtained:

$$C_1 = V_t\,/(g\,T) \;\text{ and }\; C_2 = 0\;.$$

By returning the obtained expressions for constants C_1 and C_2 into the expressions (15) and (16), the final expressions for ψ and $\dot{\psi}$ are obtained as:

$$\psi = V_t\,[\cos(k\,t) - 1]\,/(g\,T)\;; \tag{17}$$

$$\dot{\psi} = -V_t\,k \sin(k\,t)\,/(g\,T)\;; \tag{18}$$

Equalling the left side of the equation (17) to zero (inclination of rope at the end of the braking period should be equal to zero), the solutions for which the inclination of rope is equal to zero are obtained.

$$\psi = 0\;;\Rightarrow \cos(k\,t) = 1\;;\Rightarrow\; k\,t = 0,\,2\pi,...$$

The second solution is adopted ($t=2\cdot\pi/k$), which means that for $t = 2\pi\sqrt{L\,/\,g}$, the rope inclination will be equal to zero, and the velocity of the mathematical pendulum point of suspension will be also equal to zero. As the adopted solution is at the same time the solution of equation (18) i.e. $\dot{\psi}(2\pi\sqrt{L\,/\,g}) = 0$, the angular velocity of the rope inclination will also be equal to zero. On the basis of above presented facts, the duration of the braking period during the adaptive damping of mathematical pendulum oscillations with constant length and with the point of suspension moving with constant deceleration is:

$$T = 2\,\pi\sqrt{L\,/\,g}\;; \tag{19}$$

It is noticed that in this case the duration of the braking period is equal to the period of natural oscillations of the mathematical pendulum of length L.

b) *Digital damping of oscillations of the mathematical pendulum with constant rope length* [4]

During the digital damping of the oscillations there are, as it was mentioned before, three stages of motion. Constants "c_i" ($\ddot{z}_i(t) = c_i$, i=1,2,3) for boundary conditions defined by expressions (7) (braking) have, for each stage, the following values:

$$c_1 = V_t\,/(2\,t_{\mathrm{I}})\;;\; c_2 = 0\;; c_3 = V_t\,/(2\,t_{\mathrm{III}})\,,$$

where t_{I}, t_{II} and t_{III} - times of duration of the first, the second and the third stage of braking period.

Initial conditions $\psi(0)$ and $\dot{\psi}(0)$ on the basis of which constants C_{1i} and C_{2i} (i=1,2,3) for each stage are calculated are:

- the first stage:

$$\psi_{\mathrm{I}}(0)=0 \;\text{ and }\; \dot{\psi}_{\mathrm{I}}(0)=0\;;$$

- the second stage:

$$\psi_{\mathrm{II}}(0)=\psi_{\mathrm{I}k} \;\text{ and }\; \dot{\psi}_{\mathrm{II}}(0)=\dot{\psi}_{\mathrm{I}k}\;;$$

where $\psi_{\mathrm{I}k}$ and $\dot{\psi}_{\mathrm{I}k}$ - angle of rope inclination and angular velocity of rope inclination at the end of the first stage of the braking period.

- the third stage:

$$\psi_{\mathrm{III}}(0)=\psi_{\mathrm{II}k} \;\text{ and }\; \dot{\psi}_{\mathrm{III}}(0)=\dot{\psi}_{\mathrm{II}k}\;;$$

where $\psi_{\mathrm{II}k}$ and $\dot{\psi}_{\mathrm{II}k}$ - angle of rope inclination and angular velocity of rope inclination at the end of the second stage of the braking period.

By substituting the values of constants "c_i" and corresponding initial conditions in expressions (15) and (16), (procedure for each stage is identical to the one shown when defining the duration of braking period for the adaptive damping of oscillations), and by composing the obtained expressions and their equalisation to zero, (inclination and angular velocity at the end of the braking period must be equal to zero), we obtain the system of two transcendent equations from which the time of duration of each particular stage should be determined. The obtained system has the following form:

$$\begin{aligned} &K\cos(kt_I + k\,t_{II} + k\,t_{III}) - K\cos(k\,t_{II} + k\,t_{III}) \\ &+ \cos(k\,t_{III}) - 1 = 0 \\ &K\sin(k\,t_I + k\,t_{II} + k\,t_{III}) - K\sin(k\,t_{II} + k\,t_{III}) \\ &+ \sin(k\,t_{III}) = 0 \end{aligned} \tag{20}$$

where $K = t_{III} / t_I$ - relation between the assumed times of duration of the first and the third stage (constant).

Because we have the system of two equations with three unknowns, we will assume that $t_I = t_{II} = t_{III} = t$, from where follows that K=1, and we shall introduce the replacement $k{\cdot}t = \alpha$. After these assumptions and replacements the expressions (20) take the following form:

$$\begin{aligned} &\cos(3\,\alpha) - \cos(2\,\alpha) + \cos\alpha - 1 = 0 \\ &\sin(3\,\alpha) - 2\sin(2\,\alpha) + \sin\alpha = 0 \end{aligned} \tag{21}$$

Using the features of trigonometric functions, expressions (21) are transformed to the following form:

$$2\cos\alpha\left(2\cos^2\alpha - \cos\alpha - 1\right) = 0\,; \tag{22}$$

$$4\sin\alpha\cos\alpha\,(\cos\alpha - 1) = 0\,; \tag{23}$$

Solutions of equation (22) are for: cosα=0, $\Rightarrow$ $\alpha=\pi/2$, $3\pi/2$,...; for cosα=1, $\Rightarrow$ α=0, 2π,...; and for cosα=0.5, $\Rightarrow$ $\alpha=\pi/3$, $2\pi/3$, ...

Solutions of equation (23) are for: sinα=0, $\Rightarrow$ α=0, π, ...; for cosα=0, $\Rightarrow$ $\alpha=\pi/2$, $3\pi/2$,...; and for cosα=1, $\Rightarrow$ α=0, 2π, ...

The common solutions of equations (22) and (23) are for cosα=0 and cosα=1, because the duration of braking period at digital damping of oscillations is shorter than the period of natural oscillations of the mathematical pendulum (figure 2), so the only satisfactory solution (except the trivial one α=0) for the given limit is: $\alpha=\pi/2$. By reverting we obtain:

$$t = \pi/2\,k \Rightarrow t_I = t_{II} = t_{III} = (\pi/2)\sqrt{L/g}\ ;$$

The braking period (T) which meets all required conditions, at digital damping of oscillations of the mathematical pendulum with constant length, is equal to the total of times of duration of particular stages:

$$T = t_I + t_{II} + t_{III} = (3\,\pi/2)\sqrt{L/g}\ ; \tag{24}$$

which represents 3/4 of the period of natural oscillations of the mathematical pendulum of length L.

c) *Optimal damping of oscillations of the mathematical pendulum with constant rope length*

Expressions for the rope inclination, rope angular velocity, velocity of the mathematical pendulum point of suspension (crane trolley) and control obtained by solving the system of differential equations (13) for L=const. have the following forms, respectively:

$$\begin{aligned} y_1 = \psi = \\ &= C_1 e^{at}\left[(a - g\frac{x_1}{L})\cos(bt) + (-b - g\frac{x_2}{L})\sin(bt)\right] + \\ &+ C_2 e^{at}\left[(b + g\frac{x_2}{L})\cos(bt) + (a - g\frac{x_1}{L})\sin(bt)\right] + \\ &+ C_3 e^{-at}\left[(-a + g\frac{x_1}{L})\cos(bt) + (-b - g\frac{x_2}{L})\sin(bt)\right] + \\ &+ C_4 e^{-at}\left[(b + g\frac{x_2}{L})\cos(bt) + (-a + g\frac{x_1}{L})\sin(bt)\right] - \\ &- C_5 g/L \end{aligned} \tag{25}$$

$$\begin{aligned} y_2 = \dot\psi = {} & C_1\,e^{a\cdot t}\,(\alpha\cos(b\,t) - \beta\sin(b\,t)) \\ &+ C_2\,e^{a\cdot t}\,(\beta\cos(b\,t) + \alpha\sin(b\,t)) + \\ &+ C_3\,e^{-a\cdot t}\,(\alpha\cos(b\,t) + \beta\sin(b\,t)) + \\ &+ C_4\,e^{-a\cdot t}\,(-\beta\cos(b\,t) + \alpha\sin(b\,t)) \end{aligned} \tag{26}$$

$$\begin{aligned} y_3 = \dot z = \\ &= -C_1\,e^{a\,t}\left[(x_1\gamma - x_2\delta)\frac{\cos(bt)}{L} + (x_1\delta + x_2\gamma)\frac{\sin(bt)}{L}\right] - \\ &- C_2\,e^{at}\left[(-x_2\gamma - x_1\delta)\frac{\cos(bt)}{L} + (-x_2\delta + x_1\gamma)\frac{\sin(bt)}{L}\right] - \\ &- C_3\,e^{-at}\left[(x_1\gamma - x_2\delta)\frac{\cos(bt)}{L} + (-x_1\delta - x_2\gamma)\frac{\sin(bt)}{L}\right] - \\ &- C_4\,e^{-at}\left[(x_2\gamma + x_1\delta)\frac{\cos(bt)}{L} + (-x_2\delta + x_1\gamma)\frac{\sin(bt)}{L}\right] + \\ &+ C_5\,g^2\,t/L + C_6 \end{aligned} \tag{27}$$

$$\begin{aligned} u = \ddot z = {} & -C_1\,e^{at}\left(x_1\cos(bt)/L + x_2\sin(bt)/L\right) - \\ &- C_2\,e^{a\cdot t}\left(-x_2\cos(bt)/L + x_1\sin(bt)/L\right) - \\ &- C_3\,e^{-at}\left(-x_1\cos(bt)/L + x_2\sin(bt)/L\right) - \\ &- C_4\,e^{-at}\left(-x_2\cos(bt)/L - x_1\sin(bt)/L\right) + C_5 g^2/L \end{aligned} \tag{28}$$

where

$$a = \sqrt{\frac{\sqrt{1+g^2}}{L}}\sin(0.5\arctan\frac{\sqrt{4L^2 + 4gL - 1}}{2gL - 1})\ ;$$

$$b = \sqrt{\frac{\sqrt{1+g^2}}{L}}\cos(0.5\cdot\arctan\frac{\sqrt{4L^2 + 4gL - 1}}{2gL - 1})\ ;$$

$$\alpha = (a^2 - b^2 + g/L)\,L/(g+L)\ ,$$

$$\beta = \frac{2abL}{g+L}\ ,\ \ \gamma = \frac{a}{a^2+b^2}\ ,\ \ \delta = \frac{b}{a^2+b^2}\ ,$$

$$x_1 = (\alpha - 1)\,\gamma + \beta\,\delta\ ,\ \ x_2 = (\alpha - 1)\,\delta - \beta\,\gamma\ .$$

Expressions for integration constants C_i (i=1,…,6), because of their complexity are not given in the analytical form. Numerical values of constant C_i for boundary conditions defined by expressions (7) depending on the rope length i.e. the braking period defined by expression (19) for different velocities of the point of suspension, i.e. crane trolley V_t are shown in the table 1. Numerical values of constants C_i for boundary conditions defined by expression (7) depending on rope length, i.e. the braking period defined by expression (24) for different velocities of point of suspension i.e. crane trolley V_t are shown in the table 2.

Table 1. Integration constants ($T = 2\pi \sqrt{L/g}$)

L [m]	C_i	V_t [m/s]		
		0.8333	1.05	1.3333
6	1	-0.0817522	-0.1030078	-0.1308035
	2	0.0052562	0.0066228	0.0084099
	3	0.1375141	0.1732678	0.2200226
	4	0.0081229	0.0102349	0.0129966
	5	-0.0107218	-0.0135094	-0.0171548
	6	0.8389610	1.0570909	1.3423376
7	1	-0.0791063	-0.0996740	-0.1265701
	2	0.0043189	0.0054419	0.0069103
	3	0.1299424	0.1637274	0.2079078
	4	0.0066649	0.0083977	0.0106638
	5	-0.0115672	-0.0145747	-0.0185075
	6	0.8384639	1.0564645	1.3415422
8	1	-0.0769691	-0.0969811	-0.1231506
	2	0.0036061	0.0045437	0.0057698
	3	0.1241087	0.1563769	0.1985739
	4	0.0055826	0.0070341	0.0089322
	5	-0.0123549	-0.0155672	-0.0197679
	6	0.8380908	1.0559944	1.3409453

Table 2. Integration constants ($T = 0.75 \cdot 2\pi \sqrt{L/g}$)

L [m]	C_i	V_t [m/s]		
		0.8333	1.05	1.3333
6	1	0.4374305	0.5511624	0.6998888
	2	0.5762470	0.7260713	0.9219952
	3	0.8534373	1.0753309	1.3654996
	4	0.6426107	0.8096895	1.0281772
	5	-0.0185154	-0.0233294	-0.0296247
	6	0.9636119	1.2141509	1.5417790
7	1	0.4411656	0.5558686	0.7058649
	2	0.5787715	0.7292521	0.9260344
	3	0.8412426	1.0599657	1.3459881
	4	0.6379508	0.8038180	1.0207213
	5	-0.0199677	-0.0251593	-0.0319484
	6	0.9627581	1.2130752	1.5404129
8	1	0.4441347	0.5596098	0.7106156
	2	0.5808112	0.7318221	0.9292979
	3	0.8319273	1.0482285	1.3310838
	4	0.6343155	0.7992375	1.0149047
	5	-0.0213214	-0.0268649	-0.0341142
	6	0.9621176	1.2122682	1.5393882

In the case of application of expressions (24) - (28) for boundary conditions defined by expressions (6), constants C_i from tables 1 and 2 have the same absolute values, but with opposite sign.

The results obtained by adaptive (*) and optimal damping of oscillations for: the constant length of rope L=7 m, the braking period defined by expression (19) and other parameters given in the chapter 3, are presented in the figure 3, and they are: angle of the rope inclination ψ (figure 3a), angular velocity of the rope inclination $\dot{\psi}$ (figure 3b), velocity of the point of suspension (crane trolley) $\dot{z}$ (figure 3c) and acceleration of the point of suspension of the mathematical pendulum ($\ddot{z}$) (figure 3d).

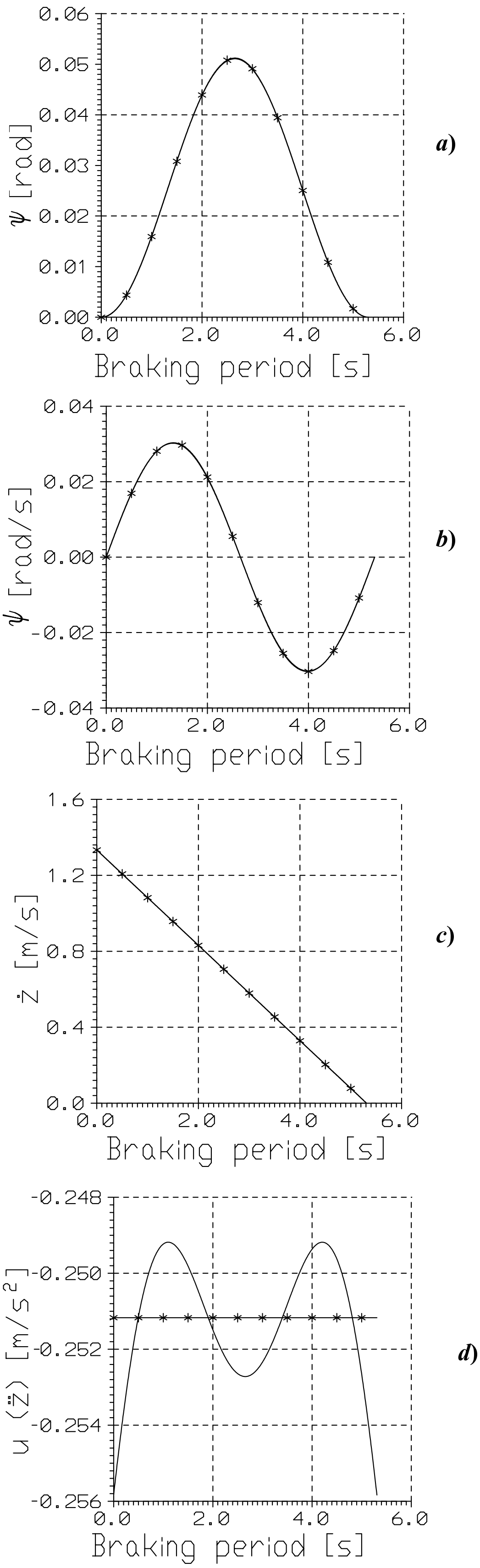

Figure 3. Change of the relevant parameters during adaptive (*) and optimal damping of oscillations.

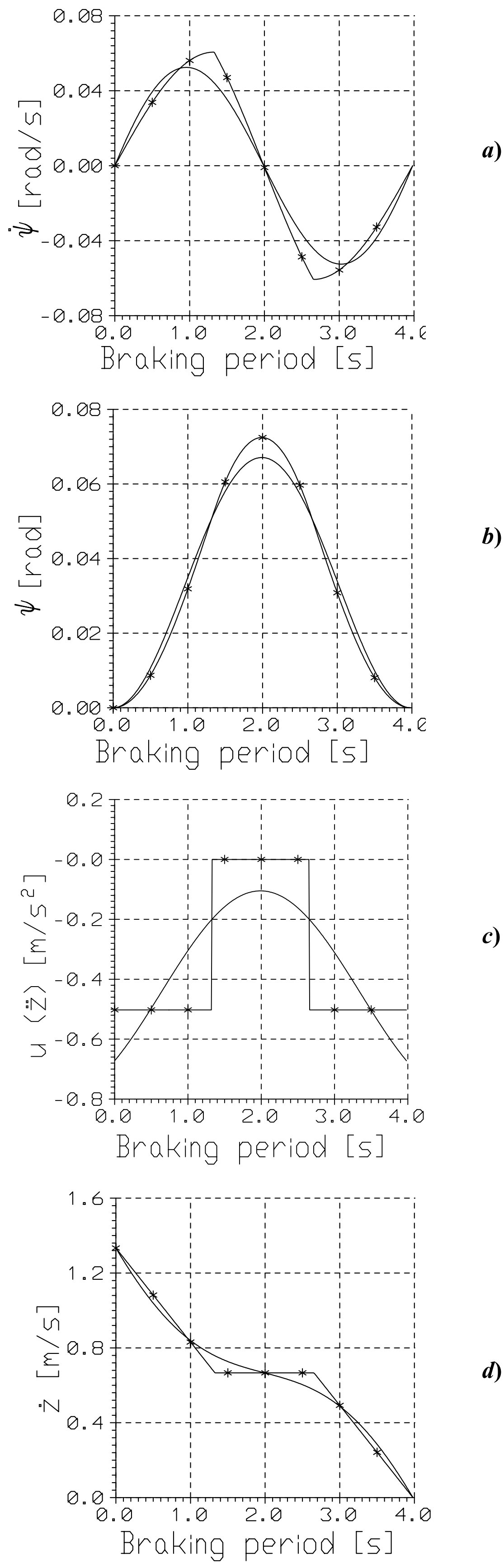

Figure 4. Change of relevant parameters during digital (*) and optimal damping of oscillations.

The results obtained by digital (*) and optimal damping of oscillations for: the constant length of rope L=7 m, the braking period defined by expression (24) and other parameters given in the chapter 3, are presented in the figure 4, and they are: angle of the rope inclination ψ (figure 4a), angular velocity of the rope inclination $\dot{\psi}$ (figure 4b), velocity of the point of suspension (crane trolley) $\dot{z}$ (figure 4c) and acceleration of the point of suspension of the mathematical pendulum ($\ddot{z}$) (figure 4d).

5. DISCUSSION OF RESULTS

During the discussion on the motion of the mathematical pendulum (chapter 4), for the optimal, adaptive and digital damping of oscillations, limitation in the sense of maximum absolute value, for the control value (acceleration / deceleration of the crane trolley) is not set. The obtained results show that the values of control for corresponding assigned velocities of the crane trolley (V_t) in stationary regimes are within recommended boundaries [7], which means that the required condition of optimality (8) provides by itself for the minimum level of control value, i.e. phase limitations of control value are not needed. It is possible to influence the maximum value of control directly by increasing (decreasing) the velocity of the point of suspension of the mathematical pendulum, i.e. crane trolley required during the acceleration (boundary conditions 6), or from which the deceleration starts (boundary conditions 7), for the assigned interval of time defined by expressions (19), (24).

Differences between adaptive and digital method of oscillations damping, in addition to different time of duration of non-stationary regimes of motion (acceleration / braking) i.e. the time of duration of unloading cycle, concern the magnitude of dynamic loads (inertial forces) which act on the carrying structure of the crane.

If the shorter time of unloading cycle is needed, i.e. if the times of non-stationary regimes are limited [7], it is necessary to use digital method of oscillations damping. This requirement causes bigger dynamic loads on the carrying structure and also higher dispersion of characteristic values ($\psi, \dot{\psi}, \dot{z}$) from the ones obtained at optimal damping of oscillations.

It is convenient to apply the adaptive method of oscillations damping if there is no limitation of the duration time of non-stationary regimes, i.e. if there is no limitation of the duration of unloading cycle, because then dynamic loads of the crane structure are equal to the optimum ones for the required conditions of oscillations damping, while the characteristic values ($\psi, \dot{\psi}, \dot{z}$) are smaller than the same obtained during the digital method of oscillations damping.

The obtained results present, which is also shown in diagrams in figures 3 and 4, that the difference between optimal method of oscillations damping and methods which are presently in practice (adaptive, digital) are minimal. In the case of load oscillation damping, (the load being considered as the mathematical pendulum of a constant length) by adaptive method, these differences practically do not exist. These facts are guiding us to the conclusion that by using the adaptive or digital method, oscillations of load during braking (acceleration) of the crane trolley can be damped in the manner very close to the optimum, so there is no need to design or develop new control systems for driving crane trolleys which would achieve the assigned motion of the trolleys at optimal damping of load oscillations.

6. CONCLUSION

Application of the obtained results is in introducing the semi-automatic unloading cycle during the bulk cargo unloading. In that case it is possible to achieve the optimal unloading cycle, dissipation of material during the grab discharging can be reduced to minimum, dynamic loads in the cranes can be smaller, and it is also possible to eliminate the influence of the human factor in the unloading process (training of operator, weather conditions, night work, etc.).

Also, obtained results i.e. analytical expressions for the rope inclination, rope angular velocity, velocity of the mathematical pendulum point of suspension (crane trolley) and control can be used as the input for oscillation damping control based on shaping inputs.

REFERENCES

[1] Auering, J. W., Simple Control Strategies for Grab Cranes to Avoid Swinging of the Load at the Destination, Förden und haben, Vol. 6, 1986, pp. 413-420 (in German).

[2] Auering, J. W., Work Speed Selection of Ship Unloading Grabbers, Förden und haben, Vol. 10, 1986, pp.713-719 (in German).

[3] Bolotnik, N. N. and Chiong, N., About optimal length of suspended load during the motion of systems based on pendulum, Izvestiya Akademii Nauk USSR - Mekhanika Tverdogo Tela, Vol. 6., 1983, pp. 28-34 (in Russian).

[4] Bugaric, U., *Contribution to Optimisation of Bulk Cargo Unloading Processes at River Ports*, M. Sc. Thesis, Faculty of Mechanical Engineering, Belgrade, 1996, p. 142. (in Serbian).

[5] Carbon, L., Advanced Drive Technology for Cranes, Bulk solids handling, Vol. 6, 1986, pp. 761-767.

[6] Chiong, N., Optimal control of motion of the pendulum based system on the surface with friction, Izvestiya Akademii Nauk USSR - Mekhanika Tverdogo Tela, Vol. 1. 198,. pp. 67-72 (in Russian).

[7] Gohberg, M., *Handbook of cranes*, Masinostroenie, Vol. 2, Leningrad, 1988, p. 559 (in Russian).

[8] Hyde, J. M. and Seering, W. P., Using Input Command Pre-Shaping to Suppress Multiple Mode Vibration, Proc. IEEE International Conference on Robotics and Automation, Sacramento, California, 1991, pp. 2604-2609.

[9] Jones,. J. F., Petterson, B. J., Oscillation Damped Movement of Suspended Objects, Proc. IEEE International Conference on Robotics and Automation, Philadelphia, Pennsylvania, 1988, pp. 956-962.

[10] Karihaloo, B. L. and Parbery, R. D., Optimal Control of a Dynamical System Representing a Gantry Crane, Journal of Optimisation Theory and Applications, Vol. 36., 1982, pp. 409-417.

[11] Kasanin, R., *Higher mathematics*, Naucna Knjiga, Vol. 1, Belgrade, 1969, p. 847 (in Serbian).

[12] Kazak S. A., *Dynamics of Bridge Cranes, Masinostroenie*, Moscow, 1968, p. 331 (in Russian).

[13] Kazak, S. A., Pendulum oscillations of load and influence of the driving mechanism to the trolley motion, Vestnik Masinostroenia, Vol. 8, 1991, pp. 30-32 (in Russian).

[14] Komarov, M. S., *Dynamics of transportation machines*, Masgiz, Moscow, 1962, p. 238 (in Russian).

[15] Kress, R. L., Jansen, J. F. and Noakes M. W., Experimental Implementation of a Robust Damped-Oscillation Control Algorithm on a Full-sized, Two-Degree-Of-Freedom, AC Induction Motor-Driven Crane, *Proc.* Fifth International Symposium on Robotics and Manufacturing, Vol. 5, Maui, Hawaii, 1994, pp. 585-592.

[16] Lobov, N. A., *Dynamics of cranes*, Masinostroenie, Moscow, 1987, p. 157 (in Russian).

[17] Moustafa, K. A. F. and Ebeid, A. M., Nonlinear Modeling and Control of Overhead Crane Load Sway, ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 110, 1988, pp. 266-271.

[18] Noakes, M. W. and Jansen, J. F., Generalized Inputs for Damped-Vibration Control of suspended payloads, Robotics and Autonomous Systems, Vol. 10, 1992, pp. 199-205.

[19] Noakes, M. W., Kress, R. L. and Appleton, G. T., Implementation of Damped-Oscillation Crane Control for Existing ac Induction Motor-Driven Cranes, Proc. Annual Meeting of the American Nuclear Society, 1993, pp. 479-485.

[20] Noakes, M. W., Petterson, B. J. and Werner, J. C., An Application of Oscillation Damped Motion for Suspended Payloads to the Advanced Integrated Maintenance Systems, Proc. Annual Meeting of the American Nuclear Society, Nashville, Tennessee, 1990, pp. 63-67.

[21] Oyler, F. J., *Handling of Bulk Solids at Ocean Ports, Stacking Blending Reclaiming*, Edited by R. H. Wohlbier, Trans Tech Publications, Clausthal, 1977, p. 863.

[22] Sage, A. P. and White, C. C., *Optimum System Control*, Prentice-Hall, Eaglewood, 1977, p. 63.

[23] Schwardtmann, K., Swing Absorber for Unloading Equipment for Bulk Goods, Förden und haben, Vol. 2, 1976, pp. 120-122 (in German).

[24] Singer, C. N. and Seering, W. P., Design and Comparison of Command Shaping Methods for Controlling Residual Vibration, Proc. IEEE International Conference on Robotics and Automation, Scottsdale, Arizona, 1989, pp. 922-927.

[25] Singer, C. N. and Seering, W. P., Preshaping Command Inputs to Reduce System Vibration, ASME Journal of Dynamic Systems, Measurement, and Control, Vol. 112, 1990, pp. 76-82.

[26] Singer, C. N. and Seering, W. P., An Extension of Command Shaping Methods for Controlling Residual Vibration Using Frequency Sampling, *Proc*. IEEE International Conference on Robotics and Automation, Nice, 1992, pp. 800-806.

[27] Sokolov, B., N., Synthesis of optimal control of pendulum energy, Izvestiya Akademii Nauk USSR - Mekhanika Tverdogo Tela, Vol. 2., 1985, pp. 54-61 (in Russian).

[28] Unbehauen, H., Metha, A. and Pura, R., On-line determination of the angle of swing angular velocity of a grab crane, Förden und haben, Vol. 6, 1987, pp. 399-403 (in German).

[29] Zaremba, A. T., Optimal pendulum motion during the phases limit of the point's suspension velocity, Izvestiya Akademii Nauk USSR - Mekhanika Tverdogo Tela, Vol. 3., 1982, pp. 28-34 (in Russian).

[30] Zrnic, Dj., Bugaric, U. and Vukovic, J., The optimisation of moving cycle of grab by unloading bridges, Proc. IFToMM 9th World Congress on the Theory of Machines and Mechanisms, Milano, Vol. 2., 1995, pp. 1001-1005.

[31] Dedijer, S., *Basics of transportation devices*, Gradjevinska knjiga, Belgrade 1983, (in Serbian).

ОПТИМАЛНО УПРАВЉАЊЕ КРЕТАЊЕМ СИСТЕМА ЗАСНОВАНИХ НА МАТЕМАТИЧКОМ КЛАТНУ КОНСТАНТНЕ ДУЖИНЕ

У. Бугарић, Ј. Вуковић

У раду се разматра премештање математичког клатна (са освртом на примену при моделирању дизаличних постројења) из стања кретања тачке вешања константном брзином у стање мировања за унапред задато време са пригушивањем осцилација на крају процеса. Решења се траже применом Понтриагин-овог принципа максимума и адаптивног односно дигиталног метода пригушивања осцилација. Као управљачка величина у оба случаја користи се убрзање тачке вешања математичког клатна. Разматран је случај са константном дужином математичког клатна.