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On Heat Conduction of a Fluid with an Internal Variable

We consider the problem of heat conduction of a fluid with an internal variable making use of rational thermodynamics with Lagrange's multipliers proposed by I. Müller. In this way we derive, among the other relations, the proportionality of entropy flux-heat flux relation.

Keywords: *heat conduction, temperature, internal variable*

1. INTRODUCTION

It is known that, for a description of a thermodynamic state in a material, it is necessary to determine the fields of density, motion and temperature. However, there exist such states that are not completely characterized by this set of the basic thermodynamic fields, but an additional field is required. This additional field is not easy to measure, and it is often called the internal variable. They have been introduced into continuum mechanics for two quite different purposes: the description of structure of the material and the characteriza-tion of the condition of the material which results from prior deformation. An other example of the internal variable is the concentration of the constituents in the mixture theory.

Note that there is no objective way to choose internal variables. The choice depends on experience, feeling or type of applications.

In order to establish a general frame for a certain class of phenomenological theories of deformations it is assumed:

1. The body can be considered as a classical continuum;
2. On the adopted level of process description the thermodynamical state of each material element is determined uniquely by the values of a set of external and internal state variables even in such cases where the body is not in thermodynamical equilibrium.

Assumption 1. implies that the material points obey the identity principle and that kinematics of the body are derivable completely from the description of the motion of the material points in a suitably defined Euclidian space of observation. Consequently the Cauchy stress tensor is symmetric.

Assumption 2. that on the adopted level of phenomenological description the knowledge of the actual values of the introduced state variables is sufficient to determine the response of the respective material element in the actual stage of any arbitrary thermodynamical process. It is unnecessary to know the previous history of the process. This means that we are dealing with so called large thermodynamical state space. Furthermore, assumption 2. implies that our phenomenological theory is local in position. Usually temperature and entropy are taken for primitive concepts. Consequently the meaning of state variables in thermodynamics is the same as in thermostatics.

The problem of heat conduction of a fluid with an internal variable was investigated by I. Müller [1], making use of rational thermodynamics with Lagrange's multipliers. Such approach often encoun-ters the problem of deriving the relation between the entropy flux and the heat flux from the conditions imposed by the entropy principle employing the general entropy inequality. This problem is usually trivial if linear constitutive relations are assumed, and otherwise it can be quite difficult even though such relation is expected. The difficulty in proving the proportionality between the entropy flux and the heat flux greatly restrains the furtherance and the sympathy of the rational thermodynamics with Lagrange multipliers. It also raises the question of whether such relation is valid in general [2].

There is an other point made by I Shih-Liu [3]: although the temperature represents a basic field in classical thermodynamics, it is not present in the basic balance equations of thermodynamics. In a systematic development of thermodynamics the absolute temperature ought to be a derived concept, as it is in classical thermostatics. Therefore, it seems natural to consider as basic fields the density of mass, velocity and the specific energy which all occur in the balance equations. This was done in a linear theory of fluid [3], with the approach of the rational thermodynamics with Lagrange multipliers. This approach has been extended to the theory of nonlinear fluid [4].

In this paper we proceed further to the problem of heat conduction of a fluid with an internal variable.

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2. EQUATIONS OF BALANCE, CONSTITUTIVE RELATIONS AND THERMODYNAMIC PROCESS

The objective of thermodynamics of a fluid with an internal variable is the determination of the following fields: density $\rho(\mathbf{x},t)$, velocity $\boldsymbol{v}(\mathbf{x},t)$, specific internal energy $\varepsilon(\mathbf{x},t)$ and specific value of internal variable $a(\mathbf{x},t)$. It is assumed that the internal variable is an objective scalar and that is additive. They are related to the equations of balance of mass, momentum, energy and internal variable:

$$\dot{\rho}+\rho v_{k,k}=0 \ , \tag{1}$$

$$\rho \dot{v}_i - t_{ik,k}=0 \ , \tag{2}$$

$$\rho\dot{\varepsilon}-t_{ik}v_{i,k}+q_{k,k}=0 \ , \tag{3}$$

$$\rho\dot{a}+q^a_{k,k}-\sigma^a=0 \ , \tag{4}$$

where t_{ik} - is symmetric stress tensor, q_k - heat flux vector, q^a_k - the flux of a and σ^a - the production density of a. The material time derivative is denoted by "$\cdot$", whereas "$,$" denotes the partial derivative with respect to Cartesian coordinates x_k of three dimensional Euclidean space E_3. For the sake of simplicity body forces, radiation supplies and the possibility of a supply to the internal variable are ignored.

In order to obtain field equation from the balance equations (1-4) the constitutive equations are needed. Following I-Shih Liu [3], the general form of constitutive equations is given by

$$S=S\left(\rho,\ v_k,\varepsilon,a,v_{k,l},\varepsilon_{,k},\ a_{,k}\right) \ ,$$

where S stands for any of dependent variables: t_{ik}, q_k, q^a_k and σ^a. The form of these constitutive equations is restricted by the principle of material objectivity, the principle of material invariance and the entropy principle. As a consequence the principle of material objectivity it appears that S cannot depend on the velocity v_k and the skew symmetric part of the gradient of velocity, i.e. $v_{[k,l]}$. Then

$$S=S\left(\rho,\ \varepsilon,\ a,\ \varepsilon_{,k},\ a_{,k},\ d_{kl}\right), \tag{5}$$

where $d_{kl}=v_{(k,l)}$ is a rate of deformation tensor or stretching tensor. This form of constitutive equations is also restricted by the principle of material invariance. As a consequence the constitutive function $S(\rho,\ \varepsilon,\ a,\ \varepsilon_{,k},\ a_{,k},\ d_{kl})$ must be isotropic functions since we are dealing with a fluid. In this way we complete the system of five field equations (1-4) which, in theory, suffice for the determination of the set of five variables $\left(\rho,v_k,\varepsilon,a\right)$. A solution of these field equations is called a *thermodynamic process in a fluid with internal variable.*

3. ENTROPY PRINCIPLE

There are several approaches to the entropy principle. One approach is that of Coleman and Gurtin [5], using Clausius-Duhem inequality. Another is the principle of Müller [6, 7], which is more general. Generally, results obtained with the two approaches are different under dynamic conditions, but the same in thermodynamic equilibrium. Here we adopted Müller's approach, which imposes further restrictions on the constitutive functions. It states that for every thermodynamic process the entropy inequality must hold

$$\rho\dot{\eta}+\phi_{k,k}\geq 0 \ , \tag{6}$$

where the specific entropy density η and the entropy flux are also, generally, given by constitutive equations of the form (5).

The constrain on the thermodynamic process can be taken into account by the use of Lagrange multipliers proposed by Liu [8]: The inequality

$$\begin{aligned}&\rho\dot{\eta}+\phi_{k,k}-\Lambda^{\rho}\left(\dot{\rho}+\rho v_{k,k}\right)-\Lambda^{v_i}\left(\rho\dot{v}_{\boldsymbol{i}}-t_{ik,k}\right)-\\&-\Lambda^{\varepsilon}\left(\rho\dot{\varepsilon}-t_{ik}v_{i,k}+q_{k,k}\right)-\Lambda^{a}\left(\rho\dot{a}+q^a_{k,k}-\sigma^a\right)\geq 0\end{aligned} \tag{7}$$

must hold for all fields $\left(\rho,v_k,\varepsilon,a\right)$ in general. The Λ^{ρ}, Λ^{v_i}, Λ^{ε} and Λ^{a}, in general are functions of all variables that occur in (5), i.e. of

$$\left(\rho,\varepsilon,a,\varepsilon_{,k},a_{,k},d_{kl}\right).$$

Insertion of the constitutive (5) (including η and ϕ_k) into the inequality (7) gives the inequality which depends linearly on

$$\left(\dot{\rho},\dot{v}_k,\dot{\varepsilon},\dot{a},\rho_{,k},\dot{\varepsilon}_{,m},\dot{a}_{,m},\dot{d}_{mm,k},\varepsilon_{,mk},a_{,mk},\ d_{mm,k}\right).$$

Since the inequality has to hold for all values of these quantities, the coefficients of these derivatives must vanish then the following conditions must hold:

$$\Lambda^{\rho}=\rho\frac{\partial\eta}{\partial\rho} \ , \tag{8}$$

$$\Lambda^{v_i}=0 \ , \tag{9}$$

$$\Lambda^{\varepsilon}=\frac{\partial\eta}{\partial\varepsilon} \ , \tag{10}$$

$$\Lambda^{a}=\frac{\partial\eta}{\partial a} \ , \tag{11}$$

$$\frac{\partial\eta}{\partial\varepsilon_{,m}}=0 \ , \tag{12}$$

$$\frac{\partial\eta}{\partial a_{,m}}=0 \ , \tag{13}$$

$$\frac{\partial\eta}{\partial d_{mn}}=0 \ , \tag{14}$$

$$\frac{\partial\phi_k}{\partial\rho}-\Lambda^{\varepsilon}\frac{\partial q_k}{\partial\rho}-\Lambda^{a}\frac{\partial q^a_k}{\partial\rho}=0 \ , \tag{15}$$

$$\left(\frac{\partial\phi_k}{\partial\varepsilon_{,m}} - \Lambda^\varepsilon \frac{\partial q_k}{\partial\varepsilon_{,m}} - \Lambda^a \frac{\partial q_k^a}{\partial\varepsilon_{,m}}\right)_{(k,m)} = 0 \ , \tag{16}$$

$$\left(\frac{\partial\phi_k}{\partial a_{,m}} - \Lambda^\varepsilon \frac{\partial q_k}{\partial a_{,m}} - \Lambda^a \frac{\partial q_k^a}{\partial a_{,m}}\right)_{(k,m)} = 0 \ , \tag{17}$$

$$\frac{\partial\phi_k}{\partial d_{mn}} - \Lambda^\varepsilon \frac{\partial q_k}{\partial d_{mn}} - \Lambda^a \frac{\partial q_k^a}{\partial d_{mn}} = 0 \, , \tag{18}$$

where round parentheses indicate symmetrization of indices. There remains the following residual inequality

$$\begin{aligned} &\left(\frac{\partial\phi_k}{\partial\varepsilon} - \Lambda^\varepsilon \frac{\partial q_k}{\partial\varepsilon} - \Lambda^a \frac{\partial q_k^a}{\partial\varepsilon}\right)\varepsilon_{,k} + \\ &+\left(\frac{\partial\phi_k}{\partial a} - \Lambda^\varepsilon \frac{\partial q_k}{\partial a} - \Lambda^a \frac{\partial q_k^a}{\partial a}\right)a_{,k} + \\ &+\left(\Lambda^\varepsilon t_{ik} - \rho\Lambda^\rho \delta_{ik}\right)d_{ik} + \Lambda^a\sigma^a \geq 0 \end{aligned} \tag{19}$$

From (12-14) we conclude that η is a function of (ρ,ε,a) only, i.e.

$$\eta = \eta(\rho,\varepsilon,a) \, . \tag{20}$$

Then from (8), (10) and (11) follows the same conclusion for Λ^ε, Λ^ρ and Λ^a. Moreover

$$\mathrm{d}\eta = \frac{1}{\rho}\Lambda^\rho \,\mathrm{d}\rho + \Lambda^\varepsilon \,\mathrm{d}\varepsilon + \Lambda^a \,\mathrm{d}a \, . \tag{21}$$

The remaining conditions left for further investigation are (15-18) and the residual inequality (19). The solutions to these restrictions, to the authors' knowledge, are not yet known.

4. TEMPERATURE AND HEAT CONDUCTION LAWS

Further on we consider the linear theory as a special, but very important, case. Then

$$t_{mn} = -p\,\delta_{mn} + \nu\, d_{kk}\,\delta_{mn} + 2\mu\, d_{mn} \ , \tag{22}$$

$$q_m = Q_\varepsilon\, \varepsilon_{,m} + Q_a\, a_{,m} \ , \tag{23}$$

$$q_m^a = P_\varepsilon\, \varepsilon_{,m} + P_a\, a_{,m} \ , \tag{24}$$

$$\sigma^a = \sigma^a(\rho,\varepsilon,a), \tag{25}$$

where the pressure p, the viscosities v and μ as well as $Q_\varepsilon, Q_a, P_\varepsilon$ and P_a are all functions of (ρ,ε,a). Also

$$\eta = \eta(\rho,\varepsilon,\, a), \tag{26}$$

$$\phi_m = R_\varepsilon\, \varepsilon_{,m} + R_a\, a_{,m} \, , \tag{27}$$

i.e. they are of the form of (22-25). Of course, R_ε and R_a are function of (ρ,ε,a). We now substitute (22-25) and (26-27) into (16) and (17) and obtain

$$\phi_k = \Lambda^\varepsilon q_k + \Lambda^a q_k^a \, , \tag{28}$$

taking into account the functional dependence of Λ^ε and Λ^a on (ρ,ε,a) only. Moreover, it is obvious that (18) is identically satisfied**.**

Next, from (28) and (15) it follows that

$$\frac{\partial\Lambda^\varepsilon}{\partial\rho} q_k + \frac{\partial\Lambda^a}{\partial\rho} q_k^a = 0 \, ,$$

Thus, whether q_k and q_k^a are collinear in any admissible thermodynamic process, which physically cannot be justified, or Λ^ε and Λ^a do not depend on ρ, i.e.

$$\Lambda^\varepsilon = \Lambda^\varepsilon(\varepsilon,a), \tag{29}$$

$$\Lambda^a = \Lambda^a(\varepsilon,a) \ . \tag{30}$$

The last restriction left for investigation is the residual inequality (19). We shall denote it by Σ and, according to Müller, [7], called it entropy production. By means of (22)-(30), we write it in explicit form as

$$\begin{aligned} \Sigma = &-\left(\Lambda^\varepsilon\, p + \rho\Lambda^\rho\right)d_{mm} + \Lambda^a\,\sigma^a + \\ &+\Lambda^\varepsilon\left(\nu\, d^2{}_{mm} + 2\mu\, d_{mn}d_{mn}\right) + \Big(\frac{\partial\Lambda^\varepsilon}{\partial\varepsilon}Q_\varepsilon + \\ &+\frac{\partial\Lambda^a}{\partial\varepsilon}P_\varepsilon\Big)\varepsilon_{,m}\,\varepsilon_{,m} + \Big(\frac{\partial\Lambda^\varepsilon}{\partial a}Q_a + \frac{\partial\Lambda^a}{\partial a}P_a\Big)a_{,m}\,a_{,m} + \\ &+\Big(\frac{\partial\Lambda^\varepsilon}{\partial\varepsilon}Q_a + \frac{\partial\Lambda^\varepsilon}{\partial a}Q_\varepsilon + \frac{\partial\Lambda^a}{\partial a}P_\varepsilon + \frac{\partial\Lambda^a}{\partial\varepsilon}P_a\Big)\varepsilon_{,m}a_{,m} \geq 0 \end{aligned} \tag{31}$$

Obviously

$$\Sigma = \Sigma\left(\rho,\varepsilon,a,\varepsilon_{,m},a_{,m},d_{mn}\right).$$

For the moment we shall consider equilibrium state of a fluid, i.e. the state defined as a process in which $(\rho, v_k, \varepsilon, a)$ are time independent and uniform. We shall denote it by the index E. Then the equations of balance of mass, momentum and energy, (1-3)**,** are identically satisfied, while the balance of the internal variable, (4), requires that σ^a vanishes. Since $\Sigma_E = 0$, one concludes that has its minimum in equilibrium, namely zero. Necessary condition for Σ to minimal in equilibrium are

$$\frac{\partial\Sigma}{\partial d_{mn}}\bigg|_E = 0 \, , \tag{32}$$

$$\frac{\partial\Sigma}{\partial a}\bigg|_E = 0 \ , \tag{33}$$

$$\frac{\partial\Sigma}{\partial\varepsilon_{,m}}\bigg|_E = 0 \, , \tag{34}$$

$$\frac{\partial\Sigma}{\partial a_{,m}}\bigg|_E = 0 \ , \tag{35}$$

as well as

$$\left\| \frac{\partial^2\Sigma}{\partial X_A \partial X_B} \right\|_E \text{ is positive semi-definite.} \tag{36}$$

Here X_A stands for one of quantities $d_{mn}, \varepsilon_{,m}, a_{,m}, a$. While (34) and (35) are identically satisfied, from (32) and (33) we obtain

$$\Lambda^{\rho}{}_{|E} = -\Lambda^{\varepsilon}{}_{|E} \frac{p_{|E}}{\rho} \quad , \tag{37}$$

$$\Lambda^{a}{}_{|E} = 0 \; . \tag{38}$$

Then from (37) and (38) we have

$$\mathrm{d}\eta_{|E} = \Lambda^{\varepsilon}{}_{|E}\mathrm{d}\varepsilon + \frac{1}{\rho}\Lambda^{\rho}{}_{|E}\mathrm{d}\rho = \Lambda^{\varepsilon}{}_{|E}\left(\mathrm{d}\varepsilon - \frac{p_{|E}}{\rho}\mathrm{d}\rho\right),$$

By comparison with Gibbs equation we conclude that

$$\Lambda^{\varepsilon}{}_{|E} = \frac{1}{T}, \tag{39}$$

where T is absolute temperature.

Further discussion is very rich but also very long (see for instance [1]).

5. CONCLUSION

In the classes of isotropic materials that involve only one constitutive vector variable [7], the assertion that entropy flux-heat flux relation $\Phi_m = \Lambda q_m$ is a straightforward proof. Using the well-known representation theorem for isotropic vector functions. In a more general case this assertion is a consequence from the assumed linear constitutive representation for the entropy flux Φ_m and the heat flux q_m [3], or a result from formidable task based on the polynomial isotropic representations. In some cases [9] the proportionality relation $\Phi_m = \Lambda\, q_m$ has been adopted without a proof and the validity of the statement is merely speculated. It is not the case in this paper since the relation (28) is derived. Moreover, this is done under the assumption, that internal energy is a basic field, taking into account two vector fields: heat flux q_m and the flux q_m^a of the internal variable a .

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О ПРОВОЂЕЊУ ТОПЛОТЕ ФЛУИДА СА УНУТРАШЊОМ ПРОМЕНЉИВОМ

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Разматра се проблем провођења топлоте флуида са унутрашњом променљивом коришћењем I. Müller-ове методе Lagrange-ових множитеља веза. На тај начин је посебно одређена релација између флукса ентропије и топлотног флукса.