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The Constructive-Graphical Stability of the Mapping Methods in the General Collinear Fields

This paper considers and analyzes the constructive-graphical stability of the mapping methods in the general collinear fields based on Laguerre's points of the absolute involution mapped. The ill-conditioned and corresponding unstable zones for this mapping method are defined and some alternative procedures for their correction are proposed. As the result of this analysis, the stabile and well-conditioned general collineation mapping methods, which can be used in the design of the projective transformations software algorithms, are created and explained. The exposed analysis is a contribution to the theory of computational and projective geometry; moreover, it makes the mapping procedures in software models of general and perspective collinear fields more accurate and effective.

Keywords*: collinear fields, involution, Laguerre's points, projective mapping, stability, well-conditioned.*

1. INTRODUCTION

Every mapping method can be characterized by the accuracy and errors caused by the imperfection of the technical instruments by which it is practically and effectively performed. All collinear mapping method use the intersections of straight lines and, without regard whether the realization of this operation is graphical or numerical, it is possible to discuss about its stability and precision, and analyze the alternative procedures which will make this projective transformation more stabile and accurate. This paper considers the constructive-graphical stability of the well known mapping methods in the collocate collinear fields based on Laguerre's points of the absolute involution mapped (the fields foci).

Despite the fact that algebraic criteria for the determination of the matrices stability are numerous and well known, analysis of the collinear mapping stability is obtained directly in this paper, by the geometrical and constructive-graphical procedures. The essential presumption of this consideration is that the instability of the mapping realization is practically caused by the extremely small intersection angle of the points radus vectors.

2. BASIC TERMINOLOGY AND DEFINITIONS

Definition 1: The method of the collinear mapping from the field P_1 to the field P_2 is stable in particular zone Ω_2 of the field P_2, if the small changes of its parameters correspond to the small shifts of the points mapped. In contrary, the method of the collinear mapping is unstable in particular zone Ω_2 of the field P_2.

Definition 2: The method of the collinear mapping from the field P_1 to the field P_2 is well-conditioned in particular area Ω_1 of the field P_1, if that mapping method is stable in the associated area Ω_2 of the field P_2. The method of the collinear mapping from the field P_1 to the field P_2 is ill-conditioned in particular area Ω_1 of the field P_1, if that mapping method is unstable in the associated area Ω_2 of the field P_2.

Definition 3: If the collocate collinear fields possess area in which some mapping method is stable, this area is the stability zone of those mapping method. The area of the collocate collinear field associated with the stability zone is well-conditioned zone of those mapping method. The ill-conditioned zones are areas in some collinear field outside of the collinear field well-conditioned zones.

If the collinear mapping method is interpreted by the algebraic equations and realized by the arithmetic operations, the conditions of the stability of that mapping method can be expressed by the algebraic language, in well known matrix form. In that sense, the mathematical concept of the matrix stability, as well as the concept of ill-conditioned and well-conditioned matrices and systems of linear equation are defined.

3. THE MEASURE OF LOCAL ABSOLUTE AND LOCAL RELATIVE ERROR

The principal parameters of the collinear mapping method based on Laguerre's points of the absolute involution mapped are: homogenous coordinates of Laguerre's points and the angular coordinates of the fields points. Let us presume that the angular

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coordinates of the points in the field P_1 are determined with some errors, and that this errors cause the errors of the angular coordinates (φ and ψ) of the mapped points in the field P_2. Also, let us presume that the absolute value of all errors is not greater then some particular number ε. It is necessary to evaluate the alteration of the points mapped position in function of this number ε.

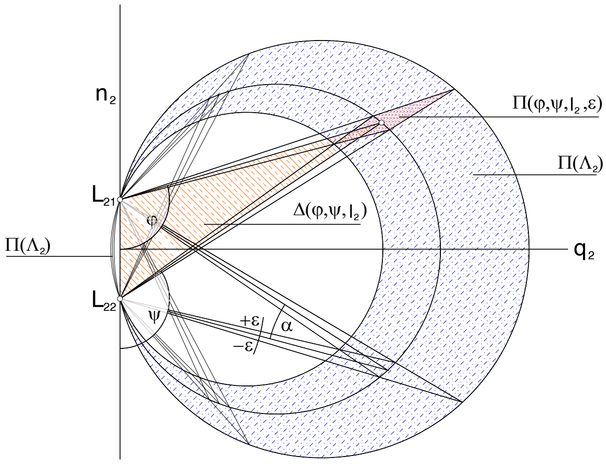

Figure 1. The measure of the local absolute and local relative error of the collinear mapping method based on fields foci

As is shown on Figure 1., the Laguerre's points (or fields foci) F_{21} and F_{22} are defined in the field P_2, as well as the set of mapped points whose radius vectors intersect in constant angle α. It is clear that this set of points belongs to the pair of equal radii circles that intersect in Laguerre's points F_{21} and F_{22}. From the fact that angular coordinates are determined with error $\pm\varepsilon$, it can be concluded that the points mapped belong to the area of the square whose sides are formed by the radus vectors with angular coordinates ($\varphi \pm \varepsilon$) and ($\psi \pm \varepsilon$). For this reason, the square areas $\Pi(\varphi, \psi, l_2, \varepsilon)$ represents the measure of local absolute error, in function of angular coordinates φ and ψ, the foci distance l_2, and error ε. The measure of the local relative error of the mapping method $\rho(\varphi, \psi, l_2, \varepsilon)$ can be represented by the ratio:

$$\rho(\varphi,\psi,l_2,\varepsilon) = \frac{\Pi(\varphi,\psi,l_2,\varepsilon)}{\Delta(\varphi,\psi,l_2)} ,$$

in which $\Delta(\varphi, \psi, l_2)$ is the area of the triangle formed by the radus vectors of the points mapped. Since the distance l_2, error ε and intersection angle α are constant in one, particularly defined, pair of collocate collinear fields, the measure of the local relative error ρ is only function of the angular coordinate φ:

$$\rho(\varphi) = \frac{\Pi(\varphi)}{\Delta(\varphi)} ; \, (\, \alpha = \text{const.} ; \; l_2 = \text{const.} ; \; \varepsilon = \text{const.} \,).$$

From the fact that this function of local relative errors is approximately constant for all $\varphi \in (0^0, 180^0)$, it can be concluded that the elliptical pencil of circles whose base points are the fields foci F_{21} and F_{22} represents the geometrical loci of nearly constant relative errors of the mapping method based on Laguerre's points of the absolute involution mapped.

4. THE MEASURE OF THE INTEGRAL MAPPING ERROR

The vertices of the squares whose area $\Pi(\varphi)$ represents the measure of local absolute error of the mapping method belongs to the same elliptical pencil of circles whose base points are fields foci F_{21} and F_{22}. As is shown on Figure 1., all mapped points belong to the circular lunettes Λ_1 i Λ_2 whose area $\Pi(\Lambda)$:

$$\Pi(\Lambda) = \Pi(\Lambda_1) + \Pi(\Lambda_2) \;,$$

decreases if intersection angle (alpha) increases. The following inequalities can be formulated from the same Figure 1.:

$$\Pi(\Lambda) \le \Pi_0 \;,\; \alpha \in [\alpha_0, (180^0 - \alpha_0)] \;,$$

$$\Pi(\Lambda) > \Pi_0 \;,\; \alpha \in (-\alpha_0, \alpha_0) \,,$$

$$\Pi_0 = \Pi_0(l_2, \varepsilon, \alpha_0) = \text{const.}$$

From this formulas, it can be concluded that area $\Pi(\Lambda)$ represents the measure of the integral error of the collinear mapping method based on the fields foci. It is essential to note that lunettes area $\Pi(\Lambda)$ can become infinitely large if intersection angle α is less than one particular value α_0. This means that error of the practical realization of the collinear mapping can be immeasurably large in those interval of intersection angles. From this considerations, one can draw the conclusion that reciprocal value $\Pi^{-1}(\Lambda)$ of area $\Pi(\Lambda)$ represents the measure of the constructive and numerical stability of the collinear mapping method based on Laguerre's points of the absolute involution mapped. Angle α_0 is critical intersection angle of the mapped points radius vectors, and its numerical value depends on quality of the technical instruments by which the collinear mapping is effectively performed.

5. THE ANALYZE DEDUCTION

The following deductions can be drawn from the exposed stability analyzes:

1. Every circle from the elliptical pencil of circles whose base points are the fields foci L_{21} and L_{22} represents the geometrical loci of constant local relative errors ρ of the numerical realisation of the mapping method based on Laguerre's points of the absolute involution mapped.

2. There are exactly two circles in the elliptical pencil whose base points are the fields foci L_{21} and L_{22} which correspond to the critical value of intersection angle of the mapped points radius vectors. This circles represent the stability zone limit of the mapping method based on the collinear fields foci, and are named as circles of critical stability for those mapping method.

3. Circles of critical stability in the collinear fields P_1 and P_2 are projectively associated to the pair of hyperbolas in the fields P_2 and P_1 respectively, which represent the hyperbolas of critical conditioned mapping method based on Laguerre's points of the absolute involution mapped.

Pair of collocate collinear fields P_1 and P_2, and hyperbolas of critical conditioned mapping method based on the fields foci are shown on Figure 2., as well as the instability zones and associated ill-conditioned areas. Those hyperbolas correspond to one particular value of critical angle α_0 and one particular value of error by which the polar coordinates of points radius vectors are determined. As is clearly shown on Figure 2., two ill-conditioned zones of the mapping method above mentioned exist in the field P_1, the first of which comprises the vanishing line r_1 and the second one the principal normal line n_1. The pair of ill-conditioned zones of the same mapping method exists in the second field too, bat they comprise vanishing line q_2 and the principal normal line n_2. The most important consequence of this considerations is that the collinear mapping method based on fields foci can not be applied in those ill-conditioned areas, and the alternative, well-conditioned mapping methods must be accomplished and performed in above mentioned zones.

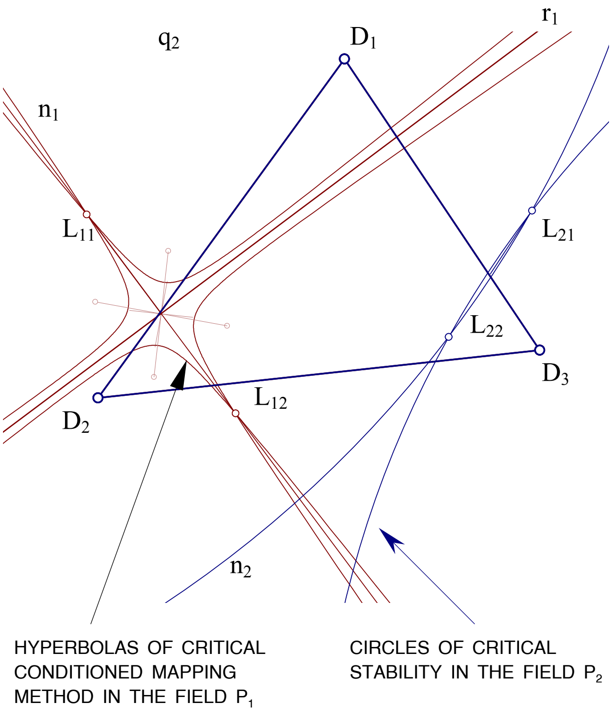

Figure 2. Hyperbolas of critical conditioned collinear mapping method based on fields foci and projectively associated circles of critical stability

5. THE ALTERNATIVE MAPPING METHODS

On Figure 3., a pair of general collinear fields, their vanishing lines r_1, q_2, and fields foci L_{11}, L_{12}, L_{21}, L_{22} on the corresponding fields principal normals n_1 and n_2 are shown. It is evident that projectively associated points A_1 and A_2 belong to the straight lines s_1 and s_2, which are respectively parallel to the vanishing line r_1, and q_2 in the collinear fields P_1 and P_2. From this facts and the theorem of the projectively associated points of the corresponding principal normal lines, the following equation can be formulated:

$$\eta \cdot \xi = l_1 \cdot l_2 / 4 \ ,$$

η, ξ – distance of the projectively associated points from the corresponding field vanishing lines,
l_1 – distance between fields foci F_{11} and F_{12},
l_2 – distance between fields foci F_{21} and F_{22} .

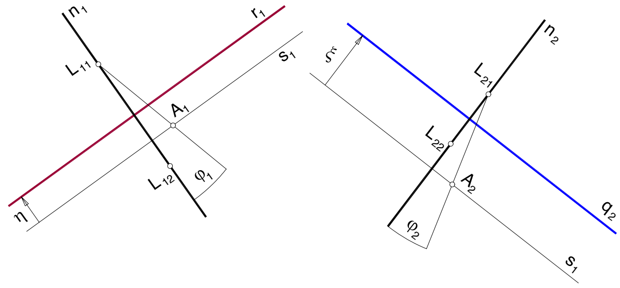

Figure 3. Alternative collinear mapping method

This hyperbolical function of coordinates η and ξ can be directly applied in the ill-conditioned zones of general and perspective collinear fields mapping method based on Laguerre's points of the absolute involution mapped. Point A_1 on the straight line s_1 that is parallel to the principal normal line n_1 is shown on Figure 2, as well as the radius vector of the point A_1 p_{11}=L_{11}-A_1, its distance η from the line n_1, and angular coordinate φ_1. The distance ξ between mapped point A_2 and vanishing line q_2, is determined in the field P_2 by the above mentioned hyperbolical function, and the position of the radius vector p_{21}, associated to the radius vector p_{11}, is found from the fact that the angular coordinates φ_2 is equal to the angular coordinate φ_1. The mapped point A_2 represents the intersection point between line s_2 and radius vector p_{21}. Since the intersection angle between lines s_2 and p_{21} is approximately equal to the right angle for the points located very close to the principal normal line n_2, this mapping method posessess a great stability in the mentioned area. This collinear transformation become unstable for the points which are extremely fare from the principal normal lines, and very close to vanishing lines. From the above, it can be concluded that this alternative mapping method possesses stability precisely in those zones in which the classical procedure, based on fields foci, is unstable.

Another collinear transformation, which can stabilize the classical mapping method, is represented on Figure 4. It is well known that the pencil of straight lines, which vertex is point O_1, is projectively transformed, from the field P_1 to the field P_2, into the pencil of parallel lines that are orthogonal to the vanishing line q_2. As is shown on Figure 4., the position of each line, which belongs to the pencil (O_1), is determined in the field P_1 by the angular coordinate φ, and the position of the corresponding line, which belongs to the projectively associated orthogonal pencil, is determined in the field P_2 by the linear coordinate ν. The following equation describes the relation between this two coordinates:

$$\nu = \text{tg}(\varphi) \cdot l_2 / 2 \ .$$

From the above, the theorem of the coordinate orthogonal net can be formulated:
The collinear transformation of the angular coordinate φ and linear coordinate η of each point in the field P_1, into the pair of linear coordinates ν and ξ of the

cooresponding point in the field P_2, is describet by function:

$$(\varphi,\eta) \to (\nu,\xi) = \left(\frac{l_2}{2} \mathrm{tg}(\varphi), \frac{l_1 l_2}{4\eta} \right) .$$

Linear coordinates ν and ξ of the mapped point represent the coordinate orthogonal net in the field P_2.

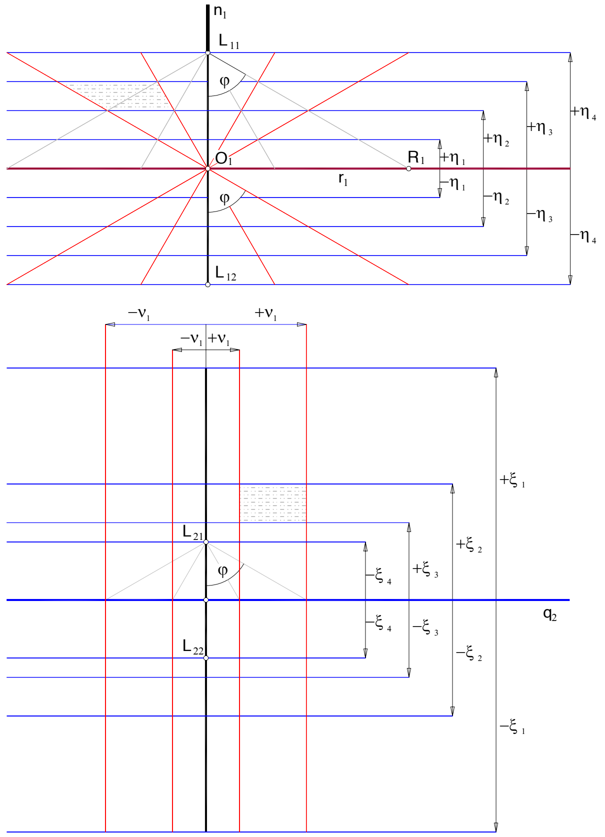

Figure 4. Coordinate orthogonal net in the general collinear fields

The collinear transformation of the angular coordinate φ and linear coordinate ξ of each point in the field P_2, into the pair of linear coordinates μ and η of the coorespon-ding point in the field P_1, is describet by function:

$$(\varphi,\xi) \to (\mu,\eta) = \left(\frac{l_1}{2} tg(\varphi), \frac{l_1 \cdot l_2}{4 \cdot \xi} \right) .$$

Linear coordinates μ and η of the mapped point represent the coordinate orthogonal net in the field P_1.

It is important to emphasize that all finite mapped points are determined in the exposed mapping procedure by the intersection of orthogonal straight lines, which means that this collinear procedure is well-conditioned and stabile for all finite points in the pair of projectively associated collinear fields. This mapping method is ill-conditioned only for infinite points:

$\varphi_1 \to 90^0$ and $\eta \to 0$ (field P_1)

$\varphi_2 \to 90^0$ and $\xi \to 0$ (field P_2)

and consequently unstabile for points which are extremely fare from the principal normal lines and venishing lines. From the above, it can be concluded that this mapping method can compensate the ill-conditioned and unstabile collinear procedures based on Laguerre's points of absolute involutions mapped.

6. CONCLUSION

This paper analyzes the stability of the mapping methods in the general and perspective collinear fields based on Laguerre's points of the absolute involution mapped. All considerations are obtained directly by the geometrical and constructive-graphical procedures. The ill-conditioned and unstable zones for this collinear transformation is defined and some alternative procedures for its correction are proposed. As the result of this analysis, the stabile and well-conditioned general collinear fields mapping methods, which can be used in computer graphics and object model design of the collinear projective transformations, are created and explained.

The exposed analysis is a contribution to the theory of computational and projective geometry; moreover, it makes the mapping procedures in software models of general and perspective collinear fields more accurate and effective.

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КОНСТРУКТИВНО ГРАФИЧКА СТАБИЛНОСТ ПОСТУПАКА ПРЕСЛИКАВАЊА У ОПШТИМ КОЛИНЕАРНИМ ПОЉИМА

Б. Попконстантиновић, З. Јели, Г. Шиниковић

Овај рад анализира конструктивно графичку стабилност поступака пресликавања у општим и перспективно колинеарним пољима која се заснивају на Лагеровим тачкама пресликаних

апсолутних инволуција. За овај метод пресликавања одређене су зоне слабе условљености и одговарајуће зоне нестабилности а, са циљем корекције, предложени су и неки алтернативни поступци пресликавања. Као резултат ове анализе, креиран је и објашњен стабилан и добро условљен метод пресликавања који се може искористити за изградњу софтверских алгоритама пројективних трансформација. Изложена анализа не представља само допринос теорији компјутерске и пројективне геометрије, већ омогућава да процедуре пресликавања у софтверским моделима опште и перспективне колинеације постану тачније и делотворније.