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Further Generalization of Golden Mean in Relation to Euler's "Divine" Equation

In this paper a new generalization of the Golden mean, as a further generalization in relation to Stakhov (1989) and to Spinadel (1999), is presented. Also, it is first observed that the Euler's "divine equation" $(a + b^n)/n = x$ *represents a possible generalization of Golden mean*

Key words: *Golden mean, symmetry, Generalized Golden mean, Euler's "divine equation"*

1. INTRODUCTION

The Golden Mean (GM) canon or ratios close to it are found in the linear proportions of masterpieces of architecture, human, animal and plant bodies. In last decades the canon is extended to the periodic system of chemical elements (*see* Luchinskiy and Trifonov, 1981; Rakočević, 1998a; Djukić and Rakočević, 2002) and to genetic code (Rakočević, 1998b), as well as to the different natural and artificial structures, especially to nanomaterial and nanotechnology (Koruga et al., 1993; Matija, 2004). As a noteworty fact, the GM is found in masterpieces of classic literature (Stakhov, 1989; Freitas, 1989; Rakočević, 2000; Rakočević, 2003).

In the present day there are minimum two generalisations of GM. First, a "vertical" generalization with x^n instead of x^2 in the equation of GM [Equations (2) and (4) in the next Section] (Stakhov, 1989); second, a "horizontal" generalization with $p > 1$ and/or $q > 1$ instead $p = q = 1$ [Equations (2) in the next Section] within a "family of mettalic means" (Spinadel, 1998; Spinadel, 1999).

2. BASIC CONCEPTS

The GM arose from the division of a unit segment line AB into two parts (Fig. 1b): first x and second $1 - x$, such that

$$\frac{x}{1-x} = \frac{1}{x} \quad . \tag{1}$$

On the other hand, one can say that GM follows from the following square equation

$$x^2 \pm px - q = 0 \,, \tag{2}$$

where $p = 1$, $q = 1$, which solutions are:

$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}\,, \quad \text{or} \quad x_{1,2} = \frac{1 \pm \sqrt{5}}{2}\,. \tag{3}$$

Stakhov (Stakhov, 1989) revealed a possible generalization of GM, from which it follows:

$$x^n + x = 1\,, \tag{4}$$

where $n = 1, 2, 3, \ldots$.

3. A NEW GENERALIZATION

In this paper we reveal, however, a further generalization, such that in equation (2) $p = 1$ and $q = m/2$; thus, we consider the following equation:

$$x^n + x = \frac{m}{2} \;, \tag{5}$$

where $n = 1, 2, 3, \ldots$; and $m = 0, 1, 2, 3, \ldots$.

For $n = 2$ and $m = 2$, we have the well-known GM (Fig. 1b), and for other coresponding values – the generalized GM's (Fig. 1a, c, d and Table 1). By this, the values of m correspond to the square roots of the odd positive integers ($r = 1, 3, 5, 7, \ldots$), through the generalized formula (3) as:

$$x_{1,2} = \frac{-1 \pm \sqrt{r}}{2}\,, \quad \text{or} \quad x_{1,2} = \frac{1 \pm \sqrt{r}}{2}\,. \tag{6}$$

In the following Fig. 1 we shall give the geometric and algebric interpretations for $m = 1, 2, 3, 4$.

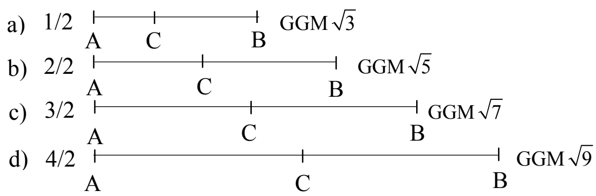

Figure 1. Generalized GM by equation (5).

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Table 1. The integer and non-integer solutions of Generalized GM

N	x_1		x_2		$\underline{h}$	m	$\sqrt{r}$		N	X_1		x_2		$\underline{h}$	M	$\sqrt{r}$
0.	0^2	+	1^2	=	**1**	0	$\sqrt{1}$		0.	0^2	+	1^2	=	**1**	0	$\sqrt{1}$
	$(0$	+	$1)^2$	=	1					$(0$	+	$1)^2$	=	1		
1.	1^2	+	2^2	=	**5**	4	$\sqrt{9}$		1.	$(x_1)^2$	+	$(x_2)^2$	=	**2**	1	$\sqrt{3}$
	$(1$	+	$2)^2$	=	9					$(x_1$	+	$x_2)^2$	=	3		
2.	2^2	+	3^2	=	**13**	12	$\sqrt{25}$		2.	$(x_1)^2$	+	$(x_2)^2$	=	**3**	2	$\sqrt{5}$
	$(2$	+	$3)^2$	=	25					$(x_1$	+	$x_2)^2$	=	5		
3.	3^2	+	4^2	=	**25**	24	$\sqrt{49}$		3.	$(x_1)^2$	+	$(x_2)^2$	=	**4**	3	$\sqrt{7}$
	$(3$	+	$4)^2$	=	49					$(x_1$	+	$x_2)^2$	=	7		
4.	4^2	+	5^2	=	**41**	40	$\sqrt{81}$		4.	1^2	+	2^2	=	**5**	4	$\sqrt{9}$
	$(4$	+	$5)^2$	=	81					$(1$	+	$2)^2$	=	9		
5.	5^2	+	6^2	=	**61**	60	$\sqrt{121}$		5.	$(x_1)^2$	+	$(x_2)^2$	=	**6**	5	$\sqrt{11}$
	$(5$	+	$6)^2$	=	121					$(x_1$	+	$x_2)^2$	=	11		
	(...)									(...)						

Consequently, it makes sense to speak about "GM per root" of 1, of 3, of 5, of 7, of 9, and so on, respectively. Also, it makes sense to see the GM as an example of "the symmetry in the simplest case" (Marcus, 1989), just in the case when $n = 1$, $m = 2$. (Notice that this case is equivalent with the case $n = 2$, $m = 4$ as it is evident from Fig. 1d).

3.1. Integer and non-integer solutions

In the following sheme we shall give the integer and non-integer solutions of Generalized GM.

From Table 1 it is evident that the sum of absolute values of solutions x_1 and x_2 to equation (5) equals is $\sqrt{r}$, which represents the first cathetus (first leg) of triangle, $\sqrt{r}-m-h$. In such a triangle, m is the second cathetus and h the hypotenuse. All such triangles on the left side in Table 1 appear as Diophantus' (Pythagorean) triangles (*see* Box 1), and on the right side their corresponding triangles. According to equations (2) and (6) there are four solutions, two positive and two negative, with two absolute values, as it is given in Table 1. Notice that $r-m-h$ triplets on the right side in Table 1 correspond to the Fibonacci triplets in first three cases (with $\underline{\boldsymbol{h}}$ as an ordinal number) (Mišić, 2004): 0−1−1, 1−2−3, 2−3−5 through a growth for Fibonacci distance triplet 1−1−2. In the next (forth) step, with the same distance 1−1−2, the Lucas' triplet 3−4−7 appears, which grows in all further steps just for one Fibonacci distance triplet 1−1−2. Notice also the next relations: on the left side in Table 1 the left-$\underline{\boldsymbol{h}}$, as well as the left-$m$, grows for 4k units (k = 0, 1, 2, 3, ...) whereas on the right side the right-$\underline{\boldsymbol{h}}$ and right-m grow just for one unit; the r on the left corresponds with r^2 on the right; the left-N$^{\text{th}}$ triangle appears in the right sequence through this "4k" regularity. (*Remark* 1: From the "4k" regularity we obtain the following triangles: 0$^{\text{th}}$-**0**$^{\text{th}}$, 1$^{\text{st}}$-**4**$^{\text{th}}$ (1$^{\text{st}}$ on the left, and **4**$^{\text{th}}$ on the right side in Table 1), 2$^{\text{nd}}$-**12**$^{\text{th}}$, 3$^{\text{rd}}$-**24**$^{\text{th}}$, 4$^{\text{th}}$-**40**$^{\text{th}}$, etc., with the next solutions: $[0+(4\times 0)=\mathbf{0}]$, $[0+(4\times 1)=\mathbf{4}]$, $[4+(4\times 2)=\mathbf{12}]$, $[12+(4\times 3)=\mathbf{24}]$, $[24+(4\times 4)=\mathbf{40}]$, etc.).

In the following example we shall consider the cases $n = 2$ and $m = 1, 2, 3$.

In the first case with $n = 2$ and $m = 1$, we have $q = 0.5$, the first case in Table 1 on the right (the first, not the zeroth) and in Fig. 1a, as the case of "GM" per $\sqrt{3}$ with the two solutions given by equations (2) and (5):

$$x_1 = \left(-1+\sqrt{3}\right)/2 = 0.3660254...$$

and

$$x_2 = \left(-1-\sqrt{3}\right)/2 = -1.3660254.... $$

The satisfactory solution is the positive solution x_1.

In the second case with $n = 2$ and $m = 2$, we have $q = 1$, the second case in Table 1 on the right and in Fig. 1b, as the case of GM per $\sqrt{5}$ wth the two solutions:

$$x_1 = \left(-1+\sqrt{5}\right)/2 = 0.6180339...$$

and

$$x_2 = \left(-1-\sqrt{5}\right)/2 = -1.6180339.... $$

In the third case with $n = 2$ and $m = 3$, we have $q = 1.5$, as the case of "GM" per $\sqrt{7}$ with the two solutions:

$$x_1 = \left(-1+\sqrt{7}\right)/2 = 0.8228756...$$

and

$$x_2 = \left(-1-\sqrt{7}\right)/2 = -1.8228756..., \text{ etc.}$$

4. THE METALLIC MEANS FAMILY

As it is known, it is very easy to find the members of "the metallic means family" (MMF) (Spinadel, 1999) as solutions of the equation (2) for various values of the parameters p and q. In fact, if $p = q = 1$, we have the GM. Analogously, for $p = 2$ and $q = 1$ we obtain the Silver mean; for $p = 3$ and $q = 1$, we get the Bronze mean. For $p = 4$; $q = 1$ we have the next metallic mean, etc. On the other hand, if $p = 1$ and $q = 2$, we obtain the Copper mean. If $p = 1$ and $q = 3$, we get the Nickel mean, and so on. Thus, we obtain all members of the MMF, which follow from square equation (2). However, by (2) and (5) we form the follow equation

$$x^n \pm px = \frac{m}{2}, \qquad (7)$$

where $n = 1, 2, 3, ...$ and $p = 1, 2, 3, ...$, then we have a generalization of MMF; furthermore, we have a unification of "vertical" and "horizontal" generalizations of GM.

Observe that De Spinadel (1999) found "the integer metallic means", for $q = 2, 6, 12, 20, 30, ...$, which solutions ($x_1$, x_2), given by equation (2), are positive integers: (1, 2), (2, 3), (3, 4), (4, 5), (5, 6) ... (Spinadel, 1999, Section 3: "Furthermore, it is very easy to verify that ... the integer metallic means, $[2,\overline{0}]$, $[3,\overline{0}]$, $[4,\overline{0}]$, ..., appear in quite a regular way"). From Table 1 it is evident that Spinadel's "integer metallic means" are related to the Diophantus' triangles (*see* Box 1), as well as to the square roots of positive integers which are squares of odd integers; thus, $r = 1, 9, 25, 49, 81, 121$, etc. On the other hand, the generalization, given by equation (5) is related to the square roots of all odd integers; thus, r = 1, 3, 5, 7, 9, 11, 13, etc.

5. THE EULER'S GENERALIZATION

In the history of mathematics it was known a conflict between the famous atheist philosopher Diderot and the famous religious mathematician Euler. ... One day Euler stepped for Diderot and stated: *"Sir, $(a + b^n)/n = x$, hence God exists; reply!"* (Eves, 1976). Diderot, as well as any one up to these days had no idea what Euler was talking about. We start here with the hypothesis (for further investigation) that Euler had the idea about "De divina proportione" of Luca Pacioli (1509) (*see* Box 2). However, after presented discussion in previous Sections of this paper we can supose that this Euler's "divine equation" can be interpreted as the most possible generalization of GM for all cases discussed in this paper. Namely, in the case $a = b$, $n = 2$ and if and only if $x = 1/2$, we have, by equations (2) and (4) just the GM; moreover, GM stands then (accordingly to the principle "if one, then all") the case of one more extended generalization: $x^n + x^{n-1} = 1$ (Stakhov, 1989, Equation 25), and/or of $x^n + px^{n-1} = m/2$, but that is the subject of a separate work.

Box 1. Diophantus' triangles

$1^2 = 0^2 + 1^2$

$5^2 = 4^2 + 3^2$

$13^2 = 12^2 + 5^2$

$25^2 = 24^2 + 7^2$

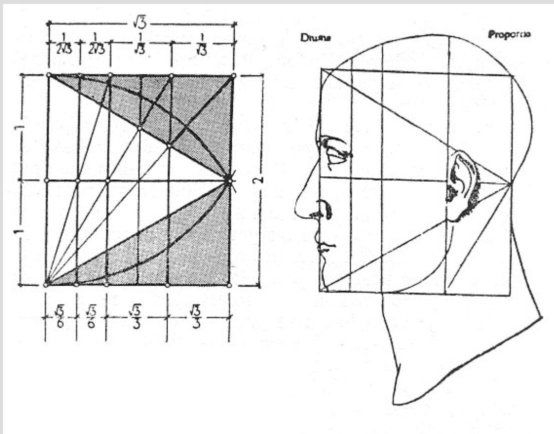
$41^2 = 40^2 + 9^2$

$61^2 = 60^2 + 11^2$

$85^2 = 84^2 + 13^2$

Diophantus' triangles are in fact the Pythagorean triangles with first cathetus "a" taken from the odd natural number series (r = 1, 3, 5, 7, 9, ...), second cathetus "b" calculated through the "4k regularity" presented in Remark 1, within Section 3.1, and hypotenuse "c", where c = b+1.

Box 2. The first Luca Pacioli's triangle



This Luca Pacioli's triangle appears as the first triangle in the right side of Table 1. The right side of Figure follows from original Pacioli's picture, while on the left side there is a Zloković's mathematical analysis (Zloković, 1955).

6. CONCLUSION

As we can see from the discussion in the previous five Sections, some known generalizations of GM, and this new one, given here, appear to be the cases of one more extended generalization, given first by Luca Pacioly, and then by Leonhard Euler. On the other hand, bearing in mind that genetic code is determined by GM (Rakočević, 1998b), one must takes answer to a question, is or is not that determination valid for Generalized GM, too. Certainly, the same question it takes set and for other natural and artificial systems.

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ДАЉА ГЕНЕРАЛИЗАЦИЈА ЗЛАТНОГ ПРЕСЕКА У РЕЛАЦИЈИ СА ОЈЛЕРОВОМ "БОЖАНСКОМ" ЈЕДНАЧИНОМ

Милоје М. Ракочевић

У раду је дата једна општија генерализација него што су "вертикална" Стаховљева (1989), унутар јединичне дужи, и "хоризонтална" госпођe Спинадел (1999) унутар скупа вишеструко умножене јединичне дужи. Ова генерализација обухвата најширу фамилију "металних пресека", као што су златни, сребрни, бронзани, бакарни, никлени, итд. Пред тога, обе наведене генерализације доведене су у везу са чувеном Ојлеровом "божанском" једначином: $(a+b^n)/n = x$.