On Brachistochronic Motion of a Multibody System with Real Constraints

Dragutin Djurić
Research Assistant
University of Belgade
Faculty of Mechanical Engineering

The paper considers a case of brachistochronic motion of the mechanical system in the field of conservative forces, subject to the action of constraints with Coulomb friction. In the special case, an analogy is made between Ashby's brachistochrone and the brachistochrone of the mechanical system with two degrees of freedom.

Keywords: *Brachistochronic motion, Coulomb friction*

1. FORMULATION OF THE PROBLEM

We are considering the motion of a mechanical system in a stationary field of potential forces where $\overline{\Pi}(\overline{q}) = \overline{\Pi}(\overline{q})$ is the system's potential energy. Let this motion be subject to the action of real constraints[1].
The configuration of the system is defined by the set of Lagrangian coordinates $\overline{q} = (\overline{q}^1, \overline{q}^2, ..., \overline{q}^n)$, to which correspond the generalized velocities $\dot{\overline{q}} = (\dot{\overline{q}}^1, \dot{\overline{q}}^2, ..., \dot{\overline{q}}^n)$. Lagrangian function of the system has the form ([2])

$$\overline{L}(\overline{q}, \dot{\overline{q}}) = \overline{T}(\overline{q}, \dot{\overline{q}}) - \overline{\Pi}(\overline{q}), \tag{1}$$

where $\overline{T}$ is kinetic energy of the system [2]

$$\overline{T} = \frac{1}{2} a_{\alpha\beta}(\overline{q})\, \dot{\overline{q}}^{\alpha} \dot{\overline{q}}^{\beta}. \tag{2}$$

Differential equations of motions of the mechanical system have a well-known form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \overline{L}}{\partial \dot{\overline{q}}^{\alpha}} - \frac{\partial \overline{L}}{\partial \overline{q}^{\alpha}} = \overline{Q}^{\mu}_{\alpha} + u_{\alpha}, \tag{3}$$

where $\overline{Q}^{\mu}_{\alpha}$ are generalized forces of Coulomb friction and u_{α} are generalized control forces.

Let us assume that initial position of the system is defined by the set of given coordinates $\overline{q}^{\alpha}_0$ at moment $t = t_0$, which is a set in advance, where it was at rest and let the final position is defined by the set of coordinates $\overline{q}^{\alpha}_1$ at moment $t = t_1$, which is unknown. The time the system needs to move from initial to final position is determined by the relation

$$I = \int_0^{t_1} \mathrm{d}t \tag{4}$$

If we assume that the system moves from initial to final configuration along one definite trajectory for which Eq. (4) has a minimum value

$$I = \int_0^{t_1} \mathrm{d}t \to \inf., \tag{5}$$

we will consider brachistochronic motion. A problem of brachistrochonic motion will be solved by variational calculus.

If we now introduce Bernoulli's condition's (cf.[3]), i.e. the conditions which do not disturb the principle of work and energy subject to the action of control forces in virtue of $u_{\alpha}\, \dot{q}^{\alpha} = 0$, we shall formulate variational problem as constrained with constraint which represents the principle of work and energy

$$\dot{\overline{T}} = \overline{P}^{\mu} - \dot{\Pi} \Rightarrow \dot{\overline{T}} + \dot{\overline{\Pi}} - \overline{P}^{\mu} = 0, \tag{6}$$

where power of generalized forces of Coulomb friction has the form

$$\overline{P}^{\mu} = \overline{Q}^{\mu}_{\alpha}\, \dot{\overline{q}}^{\alpha}, \tag{7}$$

so that relation (5) becomes

$$I_1 = \int_0^{t_1} F \mathrm{d}t \to \inf., \tag{8}$$

where

$$F(\lambda, \overline{q}, \dot{\overline{q}}, \ddot{\overline{q}}) = 1 + \lambda\left(\dot{\overline{T}} + \dot{\overline{\Pi}} - \overline{P}^{\mu}\right). \tag{9}$$

2. GENERAL PART

Let us consider a general case of brachistochronic motion presented in [5], in which Eqs. (7) has the form

$$\overline{P}^{\mu} = \overline{\psi}(\overline{q}, \dot{\overline{q}}) + \overline{\varphi}_{\beta}(\overline{q}, \dot{\overline{q}})\, \ddot{\overline{q}}^{\beta}, \tag{10}$$

assuming that

$$\overline{\psi}(\overline{q}, \dot{\overline{q}}),\ \ \overline{\varphi}_{\beta}(\overline{q}, \dot{\overline{q}}) \ \in C^2 \tag{11}$$

holds. Let us now examine a case of brachistochronic motion in which functions (11) have the following form

$$\overline{\psi}(\overline{q}, \dot{\overline{q}}) = -b_{\alpha}(\overline{q})\, \dot{\overline{q}}^{\alpha},\ \ \overline{\varphi}_{\beta}(\overline{q}, \dot{\overline{q}}) = -d_{\alpha\beta}(\overline{q})\, \dot{\overline{q}}^{\alpha}. \tag{12}$$

---

[1] Constraints with Coulomb friction.
[2] The indices take the following values: $\alpha, \beta, \gamma, \pi = 1, 2, ..., n$

Received: Decembar 2007, Accepted: Decembar 2007
Correspondence to: Dragutin Djurić
Faculty of Mechanical Engineering,
Kraljice Marije 16, 11120 Belgrade 35, Serbia
E-mail: ddj@eunet.yu

In order to avoid second-order functional let us introduce the following constraints in terms of variational calculus

$$\dot{\overline{q}}^{\alpha} - u^{\alpha} = 0,\ 2\overline{T} - a_{\alpha\beta}\, u^{\alpha}\, u^{\beta} = 0\,, \tag{13}$$

and the integrand of functional (8) gets the form[1] (cf.(6), (10) and (12))

$$F = 1 + \lambda\left(\dot{\overline{T}} + \frac{\partial\overline{\Pi}}{\partial\overline{q}^{\alpha}} u^{\alpha} - \tilde{\psi} - \tilde{\varphi}_{\beta}\,\dot{u}^{\beta}\right) + \\ +\ \theta\left(2\overline{T} - a_{\alpha\beta}\, u^{\alpha}\, u^{\beta}\right) + \sigma_{\alpha}(\dot{\overline{q}}^{\alpha} - u^{\alpha}). \tag{14}$$

where

$$\tilde{\psi}_{(\dot{\overline{q}}^{\alpha}=u^{\alpha})} = -b_{\alpha}\, u^{\alpha}, \quad \tilde{\varphi}_{\beta(\dot{\overline{q}}^{\alpha}=u^{\alpha})} = -d_{\alpha\beta}\, u^{\alpha}\,. \tag{15}$$

Assuming that conditions

$$\frac{\partial F}{\partial \overline{q}^{\alpha}} = 0, \tag{16}$$

are further satisfied, which lead to the existance of the following conditions

$$a_{\alpha\beta} = \text{const.},\ \overline{\Pi} = c_{\alpha}\,\overline{q}^{\alpha},\ c_{\alpha} = \text{const.}, \\ \frac{\partial b_{\alpha}}{\partial \overline{q}^{\gamma}} = 0,\ \frac{\partial d_{\beta\alpha}}{\partial \overline{q}^{\gamma}} = 0, \tag{17}$$

we shall apply transfomation to coordinates

$$\overline{q}^{\alpha} = k^{\alpha}_{\gamma}\, q^{\gamma}, \quad k^{\alpha}_{\gamma} = \text{const.}, \tag{18}$$

where[2]

$$\delta_{\gamma\pi} = a_{\alpha\beta}\, k^{\alpha}_{\gamma}\, k^{\beta}_{\pi}, \tag{19}$$

holds. This transformation leads to a new integrand of functional (8)

$$F^{*} = 1 + \lambda\left(\dot{T}^{*} + c^{*}_{\gamma}\,\omega^{\gamma} + b^{*}_{\gamma}\,\omega^{\gamma} + d^{*}_{\pi\gamma}\,\omega^{\pi}\,\dot{\omega}^{\gamma}\right) + \\ + \theta\left(2T^{*} - \delta_{\gamma\pi}\,\omega^{\gamma}\,\omega^{\pi}\right) + \sigma^{*}_{\gamma}\left(\dot{q}^{\gamma} - \omega^{\gamma}\right) \tag{20}$$

where

$$T^{*} = \frac{1}{2}\delta_{\gamma\pi}\,\omega^{\gamma}\omega^{\pi},\ c^{*}_{\gamma} = c_{\alpha}\, k^{\alpha}_{\gamma}, \\ b^{*}_{\gamma} = b_{\alpha}\, k^{\alpha}_{\gamma},\ d^{*}_{\pi\gamma} = d_{\alpha\beta}\, k^{\alpha}_{\pi}\, k^{\beta}_{\gamma}, \\ \dot{q}^{\gamma} = \omega^{\gamma},\ \sigma^{*}_{\gamma} = \sigma_{\alpha}\, k^{\alpha}_{\gamma}. \tag{21}$$

Formulating Euler's equations for (20) we get (cf. (21))

$$\dot{\lambda} - 2\theta = 0, \\ \sigma^{*}_{\gamma} = C^{*}_{\gamma},\ C^{*}_{\gamma} = \text{const.}, \\ \dot{\lambda}\left(\delta_{\gamma\pi} + d^{*}_{\pi\gamma}\right)\omega^{\pi} + \\ + \lambda[\left(d^{*}_{\pi\gamma} - d^{*}_{\gamma\pi}\right)\dot{w}^{\pi} - c^{*}_{\gamma} - b^{*}_{\gamma}] + C^{*}_{\gamma} = 0. \tag{22}$$

[1] $\lambda = \lambda(t)$, $\theta = \theta(t)$, $\sigma_{\alpha} = \sigma_{\alpha}(t)$ are Lagrange's multipliers.

[2] $\delta_{\gamma\pi}$ is Kroneker delta symbol.

3. MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM

Let us consider a special case of motion of mechanical system with two degrees of freedom. Assuming that condition (17) is further satisfied and having in mind (19) relations (18) get the form

$$\overline{q}^{1} = k^{1}_{1}\, q^{1} + k^{1}_{2}\, q^{2},\ \overline{q}^{2} = k^{2}_{1}\, q^{1} + k^{2}_{2}\, q^{2}, \tag{23}$$

Where

$$k^{1}_{1} = \frac{1}{\sqrt{a}},\ k^{1}_{2} = \frac{1}{\sqrt{b}}, \\ k^{2}_{1} = k^{1}_{1},\ k^{2}_{2} = s\, k^{1}_{2}, \\ a = a_{11} + 2a_{12} + a_{22},\ s = -\frac{a_{11} + a_{12}}{a_{12} + a_{22}}, \\ b = a_{11} + 2s\, a_{12} + s^{2}\, a_{22},\ a_{12} + a_{22} \neq 0. \tag{24}$$

Taking into account (23), kinetic energy of the system considered (cf. (2)) can be written in the form

$$T = \frac{1}{2}V^{2},\ V^{2} = \dot{q}^{2}_{1} + \dot{q}^{2}_{2}\,. \tag{25}$$

Potential energy (cf. (17)) gets the form

$$\Pi^{*} = c^{*}_{1}\, q^{1} + c^{*}_{2}\, q^{2}\,, \tag{26}$$

where (cf. (21), (24))

$$c^{*}_{1} = \frac{1}{\sqrt{a}}(c_{1} + c_{2}),\ c^{*}_{2} = \frac{1}{\sqrt{b}}(c_{1} + s\, c_{2})\,. \tag{27}$$

Power of generalized forces of Coulomb friction (cf. (10) obtain the form (cf. (23))

$$P^{\mu *} = -b^{*}_{i}\,\dot{q}^{i} - d^{*}_{ij}\,\dot{q}^{i}\,\ddot{q}^{j},\ i,j = 1,2, \tag{28}$$

where (cf. (21))

$$b^{*}_{1} = \frac{1}{\sqrt{a}}(b_{1} + b_{2}),\ b^{*}_{2} = \frac{1}{\sqrt{b}}(b_{1} + s\, b_{2}), \\ d^{*}_{11} = \frac{1}{a}\left(d_{11} + d_{12} + d_{21} + d_{22}\right), \\ d^{*}_{12} = \frac{1}{\sqrt{ab}}[d_{11} + d_{21} + s\,(d_{12} + d_{22})], \\ d^{*}_{21} = \frac{1}{\sqrt{ab}}[d_{11} + d_{12} + s\,(d_{21} + d_{22})], \\ d^{*}_{22} = \frac{1}{b}[d_{11} + s\,(d_{12} + d_{21}) + s^{2}\, d_{22}]. \tag{29}$$

Let us introduce natural parameter $\varepsilon = \varepsilon(t)$ (cf. [1] and [4]) by substitution

$$\dot{q}^{1} = V\cos\varepsilon, \quad \dot{q}^{2} = V\sin\varepsilon\,. \tag{30}$$

Eliminating the velocities $\dot{q}^{1}$ i $\dot{q}^{2}$ by (30), the principle of work and energy (6) has the form

$$\psi + \varphi\dot{V} + \rho V\dot{\varepsilon} = 0\,, \tag{31}$$

wherefrom integrand (9) gets the folowing form

$$F = 1+\lambda\left(\psi+\varphi\dot{V}+\rho V\dot{\varepsilon}\right)+ \\ + \sigma_1^*\left(\dot{q}^1 - V\cos\varepsilon\right)+ \sigma_2^*\left(\dot{q}^2 - V\sin\varepsilon\right) \quad (32)$$

where

$$\begin{aligned}
\psi &= r_1\cos\varepsilon + r_2\sin\varepsilon,\\
\varphi &= 1+d_{22}^* + r_4(\cos\varepsilon)^2 + \frac{r_3}{2}\sin 2\varepsilon,\\
\rho &= -d_{21}^* + r_3(\cos\varepsilon)^2 - \frac{r_4}{2}\sin 2\varepsilon,\\
r_1 &= b_1^* + c_1^*,\ r_2 = b_2^* + c_2^*,\\
r_3 &= d_{12}^* + d_{21}^*,\ r_4 = d_{11}^* - d_{22}^*.
\end{aligned} \quad (33)$$

Formulating Euler's equations for (32) in relation to q_1, q_2, V i ε, we get (cf. (21))

$$\begin{aligned}
&\dot{\sigma}_1^* = 0 \to \sigma_1^* = C_1^* = \text{const.},\\
&\dot{\sigma}_2^* = 0 \to \sigma_2^* = C_2^* = \text{const.},\\
&\vartheta_2 + (\varphi' - \rho)\lambda\dot{\varepsilon} + \varphi\dot{\lambda} = 0,\\
&(\vartheta_1 - \rho\dot{\lambda})V + \lambda[\dot{V}(\varphi'-\rho)+\psi'] = 0,
\end{aligned} \quad (34)$$

where (cf. (33))[3]

$$\begin{aligned}
\vartheta_1 &= C_1^*\sin\varepsilon - C_2^*\cos\varepsilon,\\
\vartheta_2 &= C_1^*\cos\varepsilon + C_2^*\sin\varepsilon,\\
\psi' &= r_2\cos\varepsilon - r_1\sin\varepsilon,\\
\varphi' &= r_3\cos 2\varepsilon - r_4\sin 2\varepsilon,\\
\rho' &= -r_4\cos 2\varepsilon - r_3\sin 2\varepsilon.
\end{aligned} \quad (35)$$

The condition of transversality at the right end-point gets the form

$$[1-\vartheta_2 V + \lambda\psi]_{(t=t_1)} = 0\,. \quad (36)$$

Taking into consideration that V and ε is not prescribed in the final position of the system, we have also the end-conditions

$$[\frac{\partial F}{\partial \dot{V}}]_{(t=t_1)} = 0,\ [\frac{\partial F}{\partial \dot{\varepsilon}}]_{(t=t_1)} = 0, \quad (37)$$

wherefrom we get

$$\lambda_{(t=t_1)} = \lambda_1 = 0. \quad (38)$$

As the integrand F does not contain t explicitely, Euler's equations (34) have the first integral

$$1-\vartheta_2 V + \lambda\psi = C,\ C = \text{const.} \quad (39)$$

Eliminating $\dot{V}$ by (31), Euler's equation (cf. (34)) in relation to ε gets the form

$$V\varphi\left(\vartheta_1 - \rho\dot{\lambda}\right) + \lambda[\left(\psi + V\rho\dot{\varepsilon}\right)\left(\rho-\varphi'\right)+\varphi\psi'] = 0, \quad (40)$$

wherefrom we get

$$\dot{\lambda} = \frac{1}{\rho}\{\vartheta_1 + \frac{\lambda}{V\varphi}[\left(\psi + V\rho\dot{\varepsilon}\right)\left(\rho-\varphi'\right)+\varphi\psi']\}. \quad (41)$$

If we now eliminate $\dot{\lambda}$ in Euler's equation (cf. (34)) in relation to V by (41) we shall get

$$V = \frac{\psi\left(\rho-\varphi'\right)+\varphi\psi'}{\vartheta_1\varphi\psi + \vartheta_2[\left(2\rho-\varphi'\right)\psi+\varphi\psi']}, \quad (42)$$

wherefrom in a case that the system in the start position has a velocity of known intensity $V(t_0) = 0$ we shall get (cf. (33)) a value of the natural parameter at moment $t = t_0$

$$\varepsilon_0 = \arctan[\frac{-r_2\left(1+d_{11}^*\right)+r_1 d_{21}^*}{r_2 d_{12}^* - r_1\left(1+d_{22}^*\right)}]. \quad (43)$$

If we apply the same procedure to (39), having in mind (42), we shall get the Lagrange's multiplier

$$\lambda = -\frac{\vartheta_2\rho + \vartheta_1\varphi}{\vartheta_1\varphi\psi + \vartheta_2[\left(2\rho-\varphi'\right)\psi+\varphi\psi']}, \quad (44)$$

wherefrom taking into account (38), we obtain

$$C_2^* = C_1^*\frac{\rho(\varepsilon_1)+\varphi(\varepsilon_1)\tan\varepsilon_1}{\rho(\varepsilon_1)-\varphi(\varepsilon_1)\tan\varepsilon_1} = C_1^*\frac{d_{12}^* + \kappa_1(1+d_{22}^*)}{1+d_{11}^* + d_{21}^*\kappa_1} \quad (45)$$

In order to obtain the equations of motion of the system considered (cf. (33), (35)) taking into account (45) and

$$\begin{aligned}
&\psi'' = -\psi,\ \varphi''=2\rho',\ \rho'' = -2\rho',\\
&\vartheta_1' = \vartheta_2,\ \vartheta_2' = -\vartheta_1,
\end{aligned} \quad (46)$$

the differential equations (30) can be written in the form

$$\begin{aligned}
\frac{\mathrm{d}q^1}{\mathrm{d}\varepsilon} &= -\frac{1}{(C_1^*)^2}\frac{\tilde{V}}{\psi}(\tilde{V}\rho + \varphi\tilde{V}')\cos\varepsilon,\\
\frac{\mathrm{d}q^2}{\mathrm{d}\varepsilon} &= -\frac{1}{(C_1^*)^2}\frac{\tilde{V}}{\psi}(\tilde{V}\rho + \varphi\tilde{V}')\sin\varepsilon,
\end{aligned} \quad (47)$$

where (cf. (35) , (42), (45))

$$\tilde{V} = \frac{\psi(\rho-\varphi')+\varphi\psi'}{\vartheta_3\varphi\psi + \vartheta_4(2\rho\psi - \psi\varphi' + \varphi\psi')} \quad (48)$$

$$\begin{aligned}
\tilde{V}' = &\Big\{\psi\vartheta_3\Big\{\psi\Big[2(\rho-\varphi')^2 - \varphi(\varphi+\rho')\Big]+\\
&\varphi\psi'(3\rho - 2\varphi')\ \Big\}+\\
&\vartheta_4\Big\{\big[\psi(\varphi+\rho')+\psi'\rho\big](\psi\rho' - \rho\psi') -\\
&2\psi^2\rho\psi(\varphi-\rho')\Big\}\Big\}\\
&\Big\{\vartheta_3\varphi\psi + \vartheta_4\big[\psi(2\rho-\varphi')+\varphi\psi'\big]\Big\}^{-2},
\end{aligned} \quad (49)$$

$$\begin{aligned}
\vartheta_3 &= \sin\varepsilon - \cos\varepsilon\frac{\rho(\varepsilon_1)+\varphi(\varepsilon_1)\tan\varepsilon_1}{\rho(\varepsilon_1)-\varphi(\varepsilon_1)\tan\varepsilon_1} =\\
&= \sin\varepsilon - \cos\varepsilon\frac{d_{12}^* + \kappa_1(1+d_{22}^*)}{1+d_{11}^* + d_{21}^*\kappa_1},
\end{aligned}$$

---

[3] C_1^* and C_2^* are Lagrange's multipliers corresponding to the constraints (30).

$$\vartheta_4 = \cos\varepsilon - \sin\varepsilon \frac{\rho(\varepsilon_1) + \varphi(\varepsilon_1)\tan\varepsilon_1}{\rho(\varepsilon_1) - \varphi(\varepsilon_1)\tan\varepsilon_1} =$$
$$= \cos\varepsilon - \sin\varepsilon \frac{d_{12}^* + \kappa_1(1 + d_{22}^*)}{1 + d_{11}^* + d_{21}^*\kappa_1}. \quad (50)$$

The solutions of the differential equations (47) has the form

$$q^1 = K(\Phi - \Phi_0), \ \ q^2 = K(\Psi - \Psi_0), \quad (51)$$

where (cf. (48), (49))

$$\Phi(\varepsilon_1, \varepsilon) = \int \frac{\tilde{V}}{\psi}(\tilde{V}\rho + \varphi\tilde{V}')\cos\varepsilon \ \mathrm{d}s$$
$$\Psi(\varepsilon_1, \varepsilon) = \int \frac{\tilde{V}}{\psi}(\tilde{V}\rho + \varphi\tilde{V}')\sin\varepsilon \ \mathrm{d}s \quad (52)$$
$$\Phi_0 = \Phi(\varepsilon_1, \varepsilon_0), \ \ \Psi_0 = \Psi(\varepsilon_1, \varepsilon_0),$$
$$K = \frac{1}{(C_1^*)^2}.$$

Taking into account that at final position the following relations (cf. [6])

$$\Delta(\varepsilon_1) = q^1(\Psi_1 - \Psi_0) - q^1(\Phi_1 - \Phi_0) = 0, \quad (53)$$

holds where (cf. (52))

$$\Psi_1 = \Psi(\varepsilon_1, \varepsilon = \varepsilon_1), \quad \Phi_1 = \Phi(\varepsilon_1, \varepsilon = \varepsilon_1) \quad (54)$$

we get a value of the natural parameter ε_1 at moment t_1, constant K from

$$K = \frac{q_1}{\Phi_1 - \Phi_0} = \frac{q_2}{\Psi_1 - \Psi_0}, \quad (55)$$

C_1^* (cf. (52)) and C_2^* (cf. (45)).

EXAMPLE. Let us consider motion of the mechanical system which consists of two prismatic rigid bodies and moves in homogeneous field of gravity. The configuration of the system is defined by the set of coordinates $\bar{q} = (\bar{q}^1, \bar{q}^2)$. The system starts from the position defined by coordinates $\bar{q}^1(t_0) = \bar{q}^{(10)}$ and $\bar{q}^2(t_0) = \bar{q}^{(20)}$, where it was at rest. Final position is set by $\bar{q}^1(t_1) = \bar{q}^{(11)}$ and $\bar{q}^2(t_1) = \bar{q}^{(21)}$. The coefficient of Coulomb friction on the rough inclined side (at angle α_{t0} to horizontal) of prism P_1 is μ_1. The coefficient of friction on rough horizontal plane is μ_2, (Fig 1.).

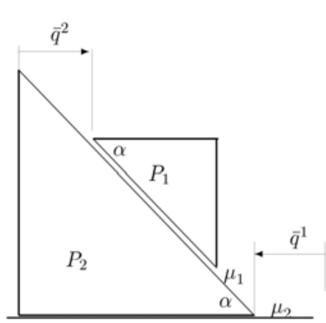

Figure 1

The differential equations of motion of the system considered (Fig. 1) have the form

$$\alpha_{11}\ddot{\bar{q}}^1 + \alpha_{12}\ddot{\bar{q}}^2 = \bar{Q}_1^\mu - \frac{\partial\bar{\Pi}}{\partial\bar{q}^1} + u_1,$$
$$\alpha_{22}\ddot{\bar{q}}^2 + \alpha_{12}\ddot{\bar{q}}^1 = \bar{Q}_2^\mu - \frac{\partial\bar{\Pi}}{\partial\bar{q}^2} + u_2, \quad (56)$$

where

$$\bar{Q}_1^\mu = -\mu_2(m + M)g - m\ddot{\bar{q}}^2 \tan\alpha,$$
$$\bar{Q}_2^\mu = -\mu_1 m(g - \ddot{\bar{q}}^1 \tan\alpha),$$
$$\frac{\partial\bar{\Pi}}{\partial\bar{q}^1} = 0, \quad \frac{\partial\bar{\Pi}}{\partial\bar{q}^2} = c_2, \quad c_2 = -mg\tan\alpha. \quad (57)$$

Relation (10) in this case have the form (cf. (57))

$$\bar{P}^\mu = \bar{\psi} + \bar{\varphi}_1\ddot{\bar{q}}^1 + \bar{\varphi}_2\ddot{\bar{q}}^2, \quad (58)$$

where

$$\bar{\psi} = -b_1\dot{\bar{q}}^1 - b_2\dot{\bar{q}}^2,$$
$$\bar{\varphi}_1 = d_{11}\dot{\bar{q}}^1 - d_{21}\dot{\bar{q}}^2,$$
$$\bar{\varphi}_2 = d_{12}\dot{\bar{q}}^1 - d_{22}\dot{\bar{q}}^2,$$
$$b_1 = g\mu_2(m + M), \quad b_2 = gm\mu_1, \quad (59)$$
$$d_{12} = -m\mu_2\tan\alpha, \quad d_{21} = -m\mu_1\tan\alpha,$$
$$d_{11} = d_{22} = 0.$$

If we now introduce relation (23), kinetic energy (25) and potential energy (26) of the system have the same form , where (cf. (27))

$$c_1^* = -\frac{gm\tan\alpha}{\sqrt{a}}, \quad c_2^* = -\frac{gms\tan\alpha}{\sqrt{b}},$$
$$a = M + m(\tan\alpha)^2,$$
$$b = M + m - 2ms + s^2 m1 + (\tan\alpha)^2, \quad (60)$$
$$s = -\frac{M}{m(\tan\alpha)^2}.$$

Power of the generalized force is presented by (28) where (cf. (29) and (60))

$$b_1^* = \frac{g\left(m\,\mu_1 + m\,\mu_2 + M\,\mu_2\right)}{\sqrt{a}},$$
$$b_2^* = \frac{g\left(m\,s\,\mu_1 + m\,\mu_2 + M\,\mu_2\right)}{\sqrt{b}},$$
$$d_{11}^* = -\frac{m\left(\mu_1 + \mu_2\right)\tan\alpha}{a},$$
$$d_{21}^* = -\frac{m\left(s\,\mu_1 + \mu_2\right)\tan\alpha}{\sqrt{a}\,\sqrt{b}}, \quad (61)$$
$$d_{12}^* = -\frac{m\left(\mu_1 + s\,\mu_2\right)\tan\alpha}{\sqrt{a}\,\sqrt{b}},$$
$$d_{22}^* = -\frac{m\,s\left(\mu_1 + \mu_2\right)\tan\alpha}{b}.$$

Taking into account (30), the relation (31) has the same form, where (cf. (33), (60))

$$\psi = g\cos\varepsilon \frac{m(\mu_1+\mu_2-\tan\alpha)+M\,\mu_2}{\sqrt{a}} +$$
$$+ g\sin\varepsilon \frac{m(s\,\mu_1+\mu_2-s\tan\alpha)+M\,\mu_2}{\sqrt{b}},$$
$$\varphi = 1 - m\sin 2\varepsilon \tan\alpha \frac{(\mu_1+\mu_2)(1+s)}{2\sqrt{a\,b}} -$$
$$- s\,m\tan\alpha \frac{\mu_1+\mu_2}{b} -$$
$$-(\cos\varepsilon)^2\, m(\mu_1+\mu_2)\tan\alpha\left(\frac{1}{a}-\frac{s}{b}\right), \qquad (62)$$
$$\rho = -m(\cos\varepsilon)^2 \tan\alpha \frac{(\mu_1+\mu_2)(1+s)}{\sqrt{a\,b}} +$$
$$+ m\tan\alpha \frac{(s\,\mu_1+\mu_2)}{\sqrt{a\,b}} -$$
$$-\frac{1}{2} m\sin 2\varepsilon \tan\alpha(\mu_1+\mu_2)\left(\frac{1}{a}-\frac{s}{b}\right).$$

Let $m=\frac{32}{13}$ denote the mass of the prism P_1 and let $M=\frac{20}{13}$ denote the mass of the prism P_2 in a suitable system of units. If $\alpha=\frac{\pi}{4}$, $\mu_1=\frac{1}{10}$ and $\mu_2=\frac{1}{50}$, then the relation (43) gives the value of natural parameter in the initial position (at moment $t=t_0$), $\varepsilon_0=-0.413771$.

If $q^1(t_1)=12$ i $q^2(t_1)=6$ (at moment $t=t_1$), then the relation (53) gives the value of the natural parameter in the final position (at moment $t=t_1$), $\varepsilon_1=0.951221$.

The graph in Fig.2 is showing the brachistochrone $q^2=f(q^1)$ of the system considered.

The changes of the power of generalized forces of Coulomb friction with respect to ε of the system considered are shown graphically in Fig.3.

After solving for

$$t_1=\int_{\varepsilon_0}^{\varepsilon_1}\frac{\mathrm{d}\varepsilon}{\dot{\varepsilon}} \qquad (45)$$

we get the time the system needs to move from the start to the final position, $t_1=1.63737s$.

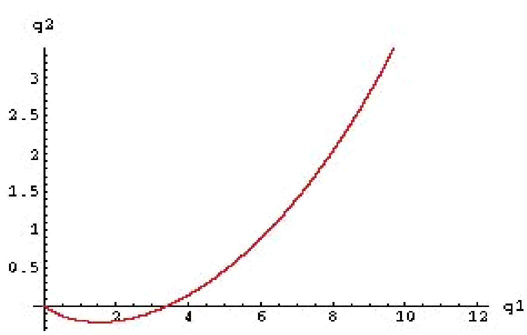

Figure 2

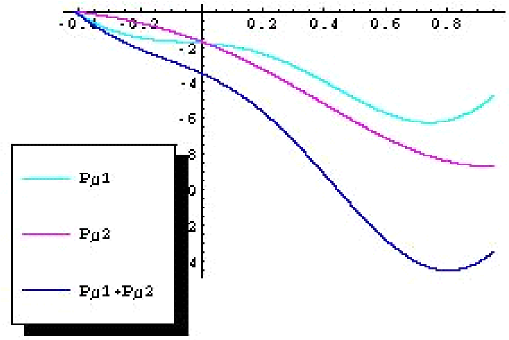

Figure 3

4. CONCLUSION

The mathematical model used to compute the brachistochrone in this special case of the multibody system with two degrees of freedom is based on variational calculus. The problem is formulated as constrained with the constraint which represents the principle of work and energy (6), where power of generalized forces of Coulomb friction has the modified form obtained from [5]. The complete analogy is made between solution obtained in an example considered and a solution in relation to material point.

REFERENCES

[1] Ashby, N. et al.: Brachistochrone with Coulomb friction, American Journal of Physics, 43, pp. 902-906, 1975.

[2] Pars, L.: *An introduction to calculus of variations*, Heinemann, London. 1962.

[3] Bernoulli, J.: Curvatura Radii in diaphanus non uniformibus, solutioque problematis a se in "Acta", 1696, p. 269 propositi de invenienda "Linea Brachistochrona" etc.", Acta Eruditorum, Leipzig, May issue, pp. 206.

[4] Čović, V., Vesković, M.: Brachistochrone on a surface with Coulomb friction, International Journal of Non-linear Mechanics, accepted paper, 2006.

[5] Čović, V., Vesković , M.: Brachistochronic motion of a multibody system with Coulomb friction, Minisymposia: Mathematical methods im Mechanics, 1st ICSSM-2007, Kopaonik, Serbia.

[6] Čović, V., Lukačević, M., Vesković , M.: On brachistochronic motions, Monographical Booklets in Applied & Computers Mathematics, PAMM, Budapest, 2007.

БРАХИСТОХРОНО КРЕТАЊЕ МЕХАНИЧКОГ СИСТЕМА СА РЕАЛНИМ ВЕЗАМА

Драгутин Ђурић

У овом раду разматрано је кретање механичког система у пољу конзервативних сила под дејством веза са Кулоновим трењем. У специјалном случају направљена је аналогија између Ешбијеве брахистохроне и брахистохроне механичког система са два степена слободе.