Milanković's Analysis of Newton's Law of Universal Gravitation

Lazar Rusov
Full Professor
University of Belgrade
Faculty of Mechanical Engineering

The first section depicts Newton's procedure for defining the law of change in Newton's universal gravitational force. Thereafter, the paper presents Milanković's analytical method of determining the law of change in the action of universal gravitational force.

Keywords: *velocity of the body (particle), deflection from the tangent, centripetal force, gravitation, sector velocity, mass of the body.*

On the occasion of the 130th anniversary of Milutin Milanković's birth, the Chair of Mechanics, Faculty of Mechanical Engineering, Belgrade calls to mind that the year 2009 is the Year of Milutin Milanković.

1. INTRODUCTION

Sir Isaac Newton (1643-1727) formulated, among other things, mathematical-mechanical model of the action of universal gravitational force in his famous 1687 publication of the *Philosophiæ Naturalis Principia Mathematica* (Mathematical principles of natural philosophy). He formulated the law that governs all material bodies of the universe: the planets, the Sun, the stars and the comets as well as the smallest particles, sand grains and air molecules. In the past, the glorification of the law went as far as the enunciation a universal law of nature, and there were attempts to interpret all natural phenomena from the standpoint of mechanics, which is certainly impossible and was not good for mechanics either.

Newton, despite being one of the creators of infinitesimal calculus, did not apply it in his analysis of the planets' orbiting the Sun, but carried out the overall analysis adhering to classical Euclidean geometry and corresponding proportions. This is the reason why the *Principia* is difficult to read even for today's specialists in mechanics.

As it comes to one of the major works on the most important law of the motion of the body that has ever been formulated in mechanics and that has enabled technological development of humanity, it is of significance to present in brief, using the improved analytical method, the entire procedure that Newton carried out to arrive at the formulation of the law of universal gravitation.

A more profound understanding of Newton's *Principia* was significantly contributed to by Alexei Nikolaevich Krylov (1863-1945), who made brilliant comments on this work and translated it into Russian [1].

In our language, Newton's law of universal gravitation was very exactly and lucidly interpreted by Milutin Milanković[1] (1879-1958) in his work "Fundamentals of Celestial Mechanics" [2], therefore it is the aim of this paper to point out to this contribution of Milanković's to an understanding of Newtonian mechanics.

[1] Milutin Milanković was professor of the subject Celestial Mechanics to the author of this work

Received: November 2009, Accepted: November 2009
Correspondence to: Dr Lazar Rusov
Faculty of Mechanical Engineering,
Kraljice Marije 16, 11120 Belgrade 35, Serbia
E-mail: zmitrovic@mas.bg.ac.rs

In Newton's time the study of mechanics was mostly oriented to the studies of celestial bodies. It was Nicolaus Copernicus (1473-1543) who published the book on planetary motion, where he pointed out that the Earth is only one of the planets and that it is not stationary but it orbits the Sun together with other planets. Danish astronomer Tycho Brahe (1546-1601) had been observing the canopy of sky for more than twenty years and every day recorded his observations on the position of the stars and the planets, the Mars in particular. Those results were made available to a German astronomer Johannes Kepler (1571-1630) and, using the data as well as his observations and appropriate analyses, Kepler formulated three outstandingly important laws of planetary motion. The first two laws were published in his work "Astronomie nova de motibus stellae Mortis" and the third one was published ten years later, 1619, in his work "Harmonices mundi". The laws read:

- I The orbit of any planet is an ellipse with the Sun at a focus.
- II The vector joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- III The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit $T_1^2 / T_2^2 = a_1^3 / a_2^3$.

Kepler's laws formulated strictly on the basis of planetary observations indicate how ingenious Kepler was. This enabled astronomers to forecast planetary position and velocity of motion accurately enough and it was being checked in practice all the time and the accuracy of the laws was confirmed, which was also later mathematically confirmed by Newtonian mechanics.

2. ELEMENTS OF NEWTON'S ANALYSIS OF THE MOTION OF THE BODY

Some elements of the analysis of body's motion will be specified, as derived by Newton. In his lengthy analysis of the motion of the body (particle), Newton associated the velocity of the body (i.e. particle) with the direction of the tangent towards the curve and it is an essential element in his analyses. In the analyses, Newton introduced the deflection of the particle from the tangent, measured to a corresponding tangent, using Lemma XI, p. 64 [1], which reads: "The arc length of a curve from its end point to the tangent drawn at the starting point of the curve (point A, Fig. 1), at infinite decrease of the arc length of a curve, for all curves in which the curvature at the point is finite, is proportional to the square of its chord", (Fig. 1).

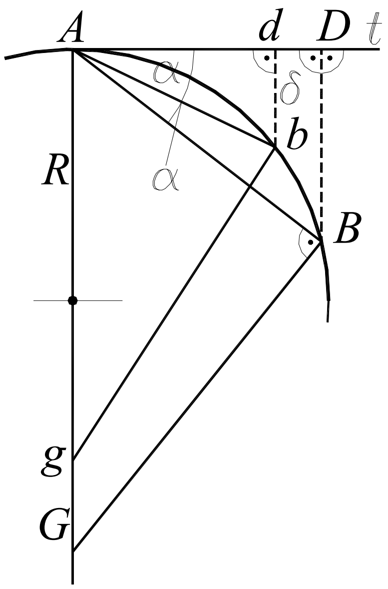

Figure 1. Deflection of the body from the tangent

From the properties of circles passing through points A, B, G and A, b, g, it follows that:

$$AB^2 = AG \cdot BD$$
$$Ab^2 = Ag \cdot bd \,. \tag{1}$$

In the above equation $bd = \delta$ is the deflection from the end point of the arc of Ab curve to the tangent, consequently:

$$bd = \delta = \frac{AB^2}{Ag} \,. \tag{2}$$

In a boundary case $Ab = s$ is the arc of a curve and as $Ag = 2R$ it follows that:

$$bd = \delta = \frac{s^2}{2R} \,. \tag{3}$$

Newton used the (3) later in his analysis of the centripetal force action. The relation (1) is arrived at by deploying trigonometric functions:

$$\frac{AB}{AG} = \frac{BD}{AB} \quad \Rightarrow \quad AB^2 = AG \cdot BD$$
$$\frac{Ab}{Ag} = \frac{bd}{Ab} \quad \Rightarrow \quad Ab^2 = Ag \cdot bd \,. \tag{4}$$

By analyzing the occurrence of centripetal force, Newton arrived at the mathematical-mechanical model of the law of universal gravitation. On page 78 [1], in theorem IV, he said: "In the motion of bodies that are moving uniformly describing different circles, centripetal forces are directed towards the centers of the circles". Furthermore, he wrote: "Since those arcs are proportional to the velocities of the bodies, (Newton wrote about the velocity of the body, but obviously it comes to the velocity of the point), centripetal forces are directly proportional to the squares of the velocities and inversely proportional to the radii of the circles".

In defining centripetal force Newton deployed determination of normal acceleration of the point (body) in its circular motion. He conducted the analysis by comparative analysis of the circular motion of the point and rectilinear motion – free fall, where the trajectory of free fall is determined by the intensity of deflection δ (3). On page 79, Corollary 9, it is written: "… arc length described by the body at any interval of time moving uniformly along the circle acted upon by centripetal force is the geometric mean of a diameter of the circle ($2R$)and a distance ($\delta = h$), raversed by the body for the same time in falling freely under the action of that force".

On the basis of Krylov's comments, using the (3), it follows:

$$s^2 = 2R\delta \quad \Rightarrow \quad 2\delta = \frac{s^2}{R} \,. \tag{5}$$

In falling freely the body traverses a distance

$$h = \delta = \frac{1}{2} a_n \tau^2 \,, \tag{6}$$

and in uniform motion along the circle the arc s is $s = V\tau$, such that

$$a_n = \frac{2\delta}{R\tau^2} = \frac{s^2}{R^2\tau^2} = \frac{V^2}{R} \,, \tag{7}$$

whereby the intensity of centripetal force is determined

$$F_\varphi = k \cdot \frac{V^2}{R} \,. \tag{8}$$

Having determined the intensity of centripetal force, Newton, in conformity with his law of action and reaction, established that "centrifugal force exerting pressure against a circular trajectory it is moving along is balanced by the opposite force that a circle pushes (pulls) the body with towards the center of a circle", (p. 81). So, the centrifugal force of action equals the centripetal force of reaction.

The next Newton's analysis is related to determining the force exerted on a body that is moving along the ellipse. In the chapter "On body motion along eccentric conical intersections", p. 91, it is written: "The body is moving along the ellipse; it is necessary to determine the law of centripetal force, directed towards the focus of the ellipse". The analysis of the body motion along the ellipse was the foundation factor in defining the law of universal gravitation later on. This is the reason why we are presenting the overall analysis that Newton carried out.

"Let S (Fig. 2.) be the focus of the ellipse. Let us place a line segment SP intersecting a diameter DK at point E and an ordinate Qv at point x determining the parallelogram $QxPR$, then EP equals longer semi-major axis of AC ellipse, for if the line HJ is running from the second focus H parallel to EC, it follows from equalities CS and CH that ES and EJ are equal, consequently:

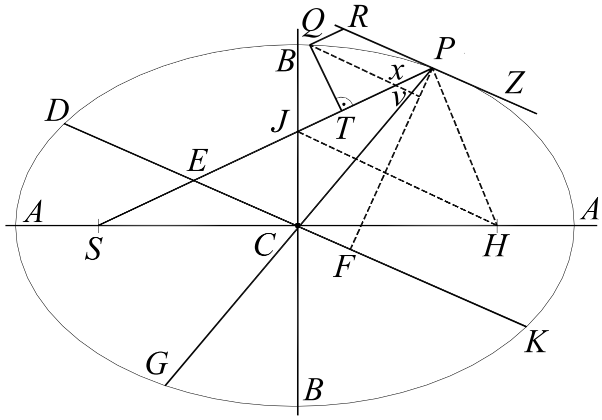

Figure 2. Determining the force when a body moves along the ellipse

$$PE = \frac{1}{2}(PS + PJ) . \quad (9)$$

As HJ is parallel to PR and angles JPR and HPZ are equal, it follows that $PJ = PH$ and the sum is $PS + PH = 2AC$. Let a perpendicular line QT be constructed on SP and let ellipse parameter be designated with L so that $L = 2\frac{BC^2}{AC}$ consequently:

$$L \cdot QR : L \cdot Pv = QR : Pv . \quad (10)$$

As $QR = Px$ it follows from the similarity of the triangles Pxv and PCE that:

$$Px : Pv = PE : PC , \quad (11)$$

it means that:

$$QR : Pv = AC : PC , \quad (12)$$

and

$$L \cdot Pv : Gv \cdot Pv = L : Gv , \quad (13)$$

and

$$Gv \cdot vP : Qv^2 = PC^2 : CD^2 . \quad (14)$$

In coinciding of the two points Q and P it will be that $Qx = Qv$, accordingly it is (Lemma XII)

$$Qx^2 : QT^2 = Qv^2 : QT^2 = EP^2 : PF^2 = \\ = AC^2 : PF^2 = CD^2 : CB^2 \quad (15)$$

consequently

$$Qv^2 : QT^2 = AC^2 : PF^2 = CD^2 : CB^2 . \quad (16)$$

When multiplying the (10), (13), (14) and (16) it is obtained:

$$L \cdot QR : QT^2 = AC \cdot L \cdot PC^2 \cdot CD^2 : PC \cdot Gv \cdot CD^2 \cdot CB^2 = \\ = 2CB^2 \cdot PC^2 \cdot CD^2 : PC \cdot Gv \cdot CD^2 \cdot CB^2 = 2PC : Gv \quad (17)$$

and in the presented position when points Q and P coincide, it will be:

$$L \cdot QR = QT^2 . \quad (18)$$

After this equation is multiplied by SP^2/QR, it is obtained

$$\frac{SP^2 \cdot QT^2}{QR} = L \cdot SP^2 \quad \Rightarrow \quad \frac{QR}{SP^2 \cdot QT^2} = \frac{1}{L \cdot SP^2} . \quad (19)$$

Accordingly (Proposition VI, Corollary 1, Corollary 5), the centripetal force is inversely proportional to $L{\cdot}SP^2$, i.e. inversely proportional to the square of the distance SP."

The derived relation (19) is Newton's answer to the question on what force must be – the law of change in force, acting upon the body, so that it moves along the elliptical trajectory. The force is inversely proportional to the square of the distance SP^2, i.e. the distance of the body from the focus of the ellipse to its position in the trajectory. This is the start of ending the question on what force is acting on the planets making them orbit the Sun along the elliptical trajectories, as Kepler established by his law and what was confirmed by astronomical observations.

Newton conducted the analysis of the motion of a body along all curves of conical intersection and thereafter he definitely approached the problem of body's motion under the action of forces of mutual attraction. In the chapter "On motion of the bodies mutually attracted by centripetal forces", p. 216, Newton wrote: "I have been presenting so far the learning about the motion of bodies attracted towards a stationary center, which hardly exists in nature. Attraction always originates from the body, and according to the third law, the action of bodies that attract and are attracted are always mutually equal, and due to this if it comes to two bodies, neither the body attracted nor the one being attracted can remain stationary, but both bodies are orbiting their common center – the focus…" "For this reason I am now moving on to setting forth the learning about the motion of bodies which are mutually attracted, considering the centripetal force as a force of attraction" (underlined by L. R.)

Newton placed centripetal force in the focus of his research on planetary motion around the Sun, which he determined by the strict analysis of the motion of the body along a circle and an ellipse. Natural phenomenon was essential to Newton, and he was looking for a method to describe it. It is quite logical, considering that bodies (planets) move along elliptical trajectories that there must exist a force making them move in such way and that force corresponds to the model of centripetal force, and on p. 219 Newton wrote: "Two bodies are mutually attracted by forces inversely proportional to the square of the distance between them, describing curves of conical intersections both around a common focus of the system and around each other, focus being the point around which they are describing those curves, and vice versa: if a body describes those curves, then the forces are inversely proportional to the square of the distance".

Analyzing the moving of the bodies that are mutually attracted, Newton defined the force of mutual action, mutual attraction as the centripetal force which is inversely proportional to the square of the distance between them and referred to the already carried out analysis of the action of centripetal force in body's motion along the elliptical trajectory, thereby defining the law of universal gravitation.

3. MILANKOVIĆ'S ANALYSIS OF NEWTON'S LAW OF UNIVERSAL GRAVITATION CARRIED OUT BY ANALYTICAL PROCEDURE

Because of the critical importance of Newtonian relation (19), Milanković demonstrated, using the improved analytical methods, that the problem of determining the motion of the body under the action of the force of shape (19) is reduced to the application of Binet's formula when the body is moving under the action of a central force. Milanković presented in detail the determination of the universal gravitational force in his work "Fundamentals of Celestial Mechanics" ([2], pp. 7-18). The basic aspects of his analysis will be displayed here.

In applying Kepler's laws to determining the planetary position, it is set out from the analytical expression for the ellipse and corresponding relations (Fig. 3):

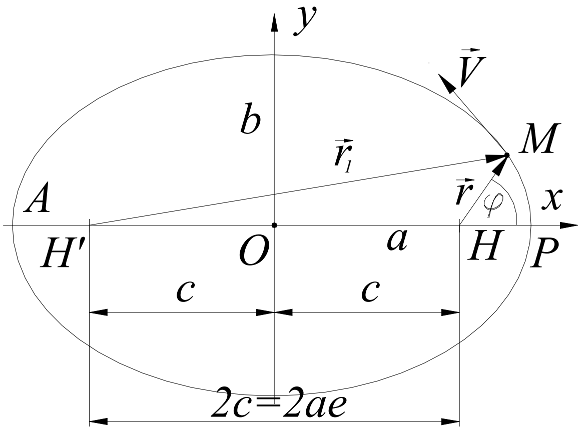

Figure 3. Determination of planet *M* position

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad r+r_1=2a$$

$$\overline{OH}=\overline{OH}=c=\sqrt{a^2-b^2}; \quad c<a . \tag{20}$$

Eccentricity of the ellipse is:

$$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}\Rightarrow a^2e^2=a^2-b^2$$

$$b^2=a^2(1-e^2)\Rightarrow b=a\sqrt{1-e^2} \tag{21}$$

and the parameter of the ellipse is:

$$p=\frac{b^2}{a}=\frac{a^2(1-e^2)}{a}=a(1-e^2) . \tag{22}$$

From the triangle *HH'M* it can be written that:

$$r_1^2=(2ae+r\cos\varphi)^2+(r\sin\varphi)^2$$

$$r_1^2=4a^2e^2+r^2+4ae\cos\varphi \tag{23}$$

considering that $r_1 = 2a - r$ it follows that:

$$4a^2-4ar+r^2=4a^2e^2+r^2+4ae\cos\varphi \tag{24}$$

that is

$$r=\frac{a(1-e^2)}{1+e\cos\varphi}=\frac{p}{1+e\cos\varphi} . \tag{25}$$

This is an equation of conical intersection in polar coordinates r and φ. For $e < 1$ the (25) determines the ellipse. Using the relation for the elliptical trajectory of the planet in polar coordinates, as well as Kepler's Second and Third law, it is possible to determine from (25) the planetary position at a given instant of time. In [2] the procedure for determining the planetary position is displayed in detail.

Newton's ingenuity is in that that he set out from the fact that there exists a natural phenomenon: the planets are orbiting the Sun along elliptical trajectories, which is accurately followed and determined by Kepler's laws. However, apart from the analysis of the trajectory, it should be responded to what must the mathematical-mechanical model of the force look like, conditioning such motion i.e. what force prevents the planet to move in a straight line and forces it all the time to move along the closed curve, elliptical trajectory. To this end, the most important task is to determine the direction and sense of the planet focus acceleration.

The position of the planet with respect to the center of the Sun, as a conditionally stationary point is $\vec{r}=r\vec{r}_0$, where $\vec{r}_0$ is unit vector. The point velocity vector in polar coordinates is determined by the expression

$$\vec{V}=\frac{d\vec{r}}{dt}=\frac{dr}{dt}\vec{r}_0+r\frac{d\varphi}{dt}\vec{p}_0=\vec{V}_r+\vec{V}_p , \tag{26}$$

and has two components, radial $\vec{V}_r$ and transverse $\vec{V}_p$ velocity.

The acceleration of the point in polar coordinates:

$$\vec{a}=\frac{d\vec{V}}{dt}=\frac{d^2\vec{r}}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\varphi}{dt}\right)^2\right]\vec{r}_0+$$

$$+\left[r\frac{d^2\varphi}{dt^2}+2\frac{dr}{dt}\frac{d\varphi}{dt}\right]\vec{p}_0=\vec{a}_r+\vec{a}_p \tag{27}$$

is also determined by: radially $\vec{a}_r$ having the direction of the position vector $\vec{r}$ and transversely $\vec{a}_p$ perpendicular to the position vector (Fig. 4).

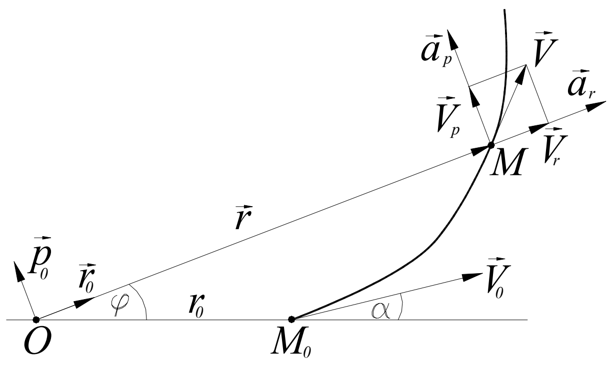

Figure 4. Determination of the planet center acceleration

Kepler established accurately that planets orbiting the Sun are not moving by uniform velocity but less fast when they are at a greater distance from the Sun and faster when they are closer to the Sun. However, the areas swept out by the planet position vector at equal intervals of time are entirely equal, which he formulated by his Second law. It is now defined that the planet sector velocity is the constant $\vec{S} = (1/2)(\vec{r} \times \vec{V}) = \text{const.}$, and in polar coordinates, respectively:

$$S = \frac{1}{2} r^2 \dot{\varphi} = \frac{1}{2} r_0^2 \dot{\varphi}_0 = \frac{1}{2} r_0 V_0 \sin\alpha = \text{const.}$$

$$r^2 \dot{\varphi} = r_0^2 \dot{\varphi}_0 = 2S = C = \text{const.} \quad (28)$$

Transverse point acceleration (the planet center) is possible to be written:

$$a_p = r\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(C), \quad (29)$$

therefore, considering (28) it follows that transverse component $\vec{a}_p$ of the planet center acceleration equals zero

$$a_p = 0. \quad (30)$$

Thus the planet center acceleration when orbiting the Sun at any instant of time i.e. at any position of the planet in elliptical trajectory has the direction of position vector $\vec{r}$, i.e. the direction joining the planet and the Sun:

$$\vec{a} = \vec{a}_r = \left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right]\vec{r}_0. \quad (31)$$

Further transformation of the (31) is done to analytically express the intensity of acceleration in a function of the intensity of the planet position vector r. To this end, it is:

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{C}{r^2} = -C\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{r}\right) \quad (32)$$

$$\ddot{r} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\dot{r}}{\mathrm{d}\varphi}\frac{C}{r^2} = \frac{C}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[-C\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{r}\right)\right] \quad (33)$$

i.e.

$$\ddot{r} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{C^2}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\left(\frac{1}{r}\right) \quad (34)$$

therefore, the planet center radial acceleration is:

$$a_r = \left[-\frac{C^2}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\left(\frac{1}{r}\right) - r\frac{C^2}{r^4}\right] = -\frac{C^2}{r^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]. \quad (35)$$

The planets are orbiting the Sun in elliptical trajectories, and using the equation of the ellipse in polar coordinates, it is found that:

$$r = \frac{p}{1 + e\cos\varphi} \quad \Rightarrow \quad \frac{1}{r} = \frac{1}{p} + \frac{e}{p}\cos\varphi$$

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{r}\right) = -\frac{e}{p}\sin\varphi, \quad \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\left(\frac{1}{r}\right) = -\frac{e}{p}\cos\varphi \quad (36)$$

therefore:

$$a_r = -\frac{C^2}{r^2}\left(-\frac{e}{p}\cos\varphi + \frac{1}{p} + \frac{e}{p}\cos\varphi\right) = -\frac{C^2}{p}\frac{1}{r^2} \quad (37)$$

i.e.

$$\vec{a} = \vec{a}_r = -\frac{C^2}{p}\frac{1}{r^2}\vec{r}_0. \quad (38)$$

Milanković derived the (38) using the improved analytical method and it is another form of the (19) that Newton had derived in analyzing the centripetal force exerted on the body moving along the elliptical trajectory. From (38) and (19), respectively, there follows an essential kinematic characteristic of the planet orbiting the Sun, and it is that the planet center acceleration is directed towards the Sun and the intensity of acceleration is inversely proportional to the square of the planet's distance from the Sun. Today it all appears to be very simple: after the planet center acceleration has been determined, a force will be obtained, multiplying by the mass of the planet. However, the term mass was not introduced into the science of mechanics before Newton's time.

The aim of Newton's further analysis was to incorporate the elements determined by Kepler's Second and Third law into defining the intensity of acceleration $\vec{a}_r$. Milanković did the same using another procedure.

In one complete planet's revolution round the Sun, the position vector sweeps out (describes) the entire area of the ellipse $A = \pi ab$. As sector velocity is a constant, it can be written that:

$$\frac{1}{2} r^2 \dot{\varphi} = \frac{A}{T} \Rightarrow r^2 \dot{\varphi} = \frac{2A}{T} = \frac{2\pi ab}{T} = C = \text{const.} \quad (39)$$

and using $b^2/a = p$, substituting in (39), it is obtained:

$$a = a_r = -\frac{C^2}{p}\frac{1}{r^2} = -\frac{4\pi^2 a^2 b^2}{pT^2}\frac{1}{r^2} =$$

$$= -\frac{4\pi^2 a^2 pa}{pT^2}\frac{1}{r^2} = -\frac{4\pi^2 a^3}{T^2}\frac{1}{r^2}. \quad (40)$$

As a^3/T^2 according to Kepler's Third law, is a constant and equal for all planets, then:

$$4\pi^2 \frac{a^3}{T^2} = \mu = \text{const.} \quad (41)$$

is also a constant for all planets and it follows that the planet center acceleration:

$$\vec{a} = \vec{a}_r = -\frac{\mu}{r^2}\vec{r}_0 \quad (42)$$

is directed towards the Sun at each of its positions. Using the (42), Milanković incorporated via a constant μ the components of Kepler's Second and Third law in defining the action of universal gravitational force.

Newton deduced that the planet center acceleration (42) is in its nature identical with the gravity force acceleration, i.e. his deduction was that attraction force of the Earth manifested in body's falling on the Earth, is transferred onto all bodies in the universe, which means onto the Moon as the Earth's satellite. There remained for him to calculate the Earth's acceleration using the already developed theory. According to the (42), acceleration decreases with the square of the distance and since the body on the surface of the Earth has acceleration of g = 9.81 m/s^2, as already calculated by G. Galilei (1564-1642) and Huygens using the pendulum, Newton started from:

$$a_r = \frac{R^2}{r^2} g \quad \Rightarrow \quad r = R, a_r = g \tag{43}$$

where r is calculated from the center of the Earth to the Moon, R is the radius of the Earth, so starting from $a_r = -\frac{4\pi^2 a^3}{T^2} \frac{1}{r^2}$, then for $r = R$, he found that:

$$a_r = g = -\frac{4\pi^2 a^3}{R^2 T^2}. \tag{44}$$

On the basis of numerical data for the Earth's radius and semi-major axis a of the Moon's orbit, adopting that it is a circle, and that the time T of a single Moon's trek around the Earth is 27 days and 7 hours, he calculated the Earth's gravity acceleration. Newton checked all his calculations by the achieved results, and even though the acceleration of the Earth's gravity had been calculated as early as 1666, he published his results 20 years later, for due to inaccurate data on the Earth's radius, his results were not in agreement with those determined experimentally by Huygens, who used the pendulum. And it was only when the Earth's radius was more accurately determined that Newton's calculations for determining the Earth's gravity acceleration approximately coincided with the experimental ones. And finally, he received the confirmation of the accuracy of his theory, so in 1686 there came out of print his capital work the *Principia*.

Since the same body depending on its position relative to the center of the Earth has different accelerations and different weights, decreasing with body's moving away from the Earth, Newton introduced the concept of body mass into mechanics, which remains constant for a given body. He defined weight as the product of mass and acceleration that is imparted to the body by a force. Introduction of the body mass m as an invariant quantity produced positive far-reaching effects on the development of mechanics, because in zero gravity the body has no weight but possesses finite mass. Today this is very evidently being accomplished in the area of astronautics.

Multiplying the acceleration (42) by the planet's mass m, the force exerted on the planet orbiting the Sun is obtained:

$$\vec{F} = -\frac{\mu m}{r^2} \vec{r}_0 . \tag{45}$$

The force is directed towards the Sun and represents the Sun's attractive force that attracts the planet. Since Newton had already laid down the principle of action and reaction, he deduced that the planet of mass m also attracts the Sun of mass M by equal force, so that force is the product of the Sun's mass and its acceleration. Introducing to the analysis:

$$f = \frac{\mu}{M} = \frac{4\pi^2 a^3}{T^2} \frac{1}{M}, \tag{46}$$

which is the constant for all planets of the solar system, and also for the Moon as the Earth's satellite, substituting $\mu = fM$ in (45) it is obtained:

$$\vec{F} = -\frac{\mu m}{r^2} \vec{r}_0 = -f \frac{mM}{r^2} \vec{r}_0, \tag{47}$$

a general expression for the universal gravitational force ([2], p. 18). The (47) holds for all material bodies of the universe and that law now reads: "All bodies attract each other by the force directly proportional to the product of their masses and inversely proportional to the square of the distance between them".

Newton outlined his law of universal gravitational force in a number of writings, especially in the analysis of the Moon's orbiting the Earth and planets' orbiting the Sun. So on p. 219 he wrote: "Two bodies mutually attracted by forces inversely proportional to the square of the distance between them are describing both around the common center and around each other the curves of conical intersection with a focus as a point around which curves are described, and vice versa: if bodies describe those curves the forces are inversely proportional to the square of the distance". On page 510, "Forces by which the major planets are constantly deflecting from rectilinear motion, and keep them retaining in their orbits, are directed towards the Sun and are inversely proportional to the square of the distance from its center". On page 512 he said: "And thus, the force retaining the Moon in its orbit, if the orbit is removed to the surface of the Earth, becomes the weight of the body, and it is the force we refer to as the weight or gravitation. For, if the weight of the body were different from gravitation, then the bodies falling on the Earth would be subjected to the action of two forces (weight and gravitation, noted by L. R.) and they would be falling twice as fast and would be describing in the first second of their falling at 30 1/6 Paris feet, which is perfectly conflicting with the experiment".

Concerning the gravitational force, Newton wrote on p. 514: "So far, we have referred to that force, retaining the celestial bodies in their orbits, as centripetal, however, as has been demonstrated it is a gravitational force, and we will keep calling it like that, for the cause of that centripetal force retaining the Moon in its orbit, according to the rules I, II and IV must be distributed to all planets," and on p. 518: "Gravitation is present on all bodies in general and is proportional to the mass of each body".

4. CONCLUSION

"Golden apple" could fall for Newton because he possessed enormous knowledge of mathematics and physics of the day and a brilliant talent for studying natural phenomena. This made possible for him to conduct versatile analysis of all results achieved by then on the motion of celestial bodies and to make such generalization, which provided for theoretical mechanics to develop in a variety of directions thus leading to the present day's development of engineering. That is why Newton's law of universal gravitation will remain to hold for all time, regardless of the continuously present unsolved secret on the cause of the gravitational force acting.

The work *Canon of Insolation and the Ice Age Problem* placed Milanković among the most prominent scientists in the world. His analysis of Newton's universal gravitational force markedly contributed to successful studies of theoretical mechanics in our country and its application to a diversity of engineering problems.

5. BIOGRAPHY

Milutin Milanković (May 28, 1879 – December 12, 1958) was born in a small place of Dalj, near Osijek, what was then part of Austro-Hungary. He graduated in Civil Engineering from the Technical High School in Vienna in 1902 and earned a doctorate in technical sciences in 1904. At age twenty-five, he became the first Serbian Doctor of Technical Sciences. He then worked for an engineering firm Betonbau-Unternehmung in Vienna. He was involved in Civil Engineering until 1909 when he was offered a "Chair of Applied Mathematics" at the Faculty of Philosophy, University of Belgrade in Serbia. There he taught rational mechanics, celestial mechanics and theoretical physics. Though he continued to pursue his investigations pertaining to the application of reinforced concrete, he decided to concentrate on fundamental research [3].

Milanković wrote a number of books and textbooks and in 1941 the printing of his great work *Canon of Insolation of the Ice-Age Problem* (Kanon der Erdbesttrahlung und siene Anwendung auf das Eiszeitenproblem) was completed, and translated into English in 1969.

Milanković made a remarkable contribution to the reform of Gregorian and Julian Calendars, proposing a common and most accurate calendar (Milanković's calendar).

Milanković was elected a corresponding member of the Serbian Academy of Arts and Sciences in 1920, a full member in 1924. He was also a member of other academies, scientific societies…

During German occupation in the Second World War Milanković, like several professors of Belgrade University, refused to sign "The Appeal to the Serbs", rejecting to support the occupation of the country.

In return for his outstanding scientific achievements the world named a crater on the Moon, a crater on the Mars and an asteroid after Milanković. European Geophysical Society established A Milutin Milanković medal for outstanding results achieved in the area of climatology and meteorology.

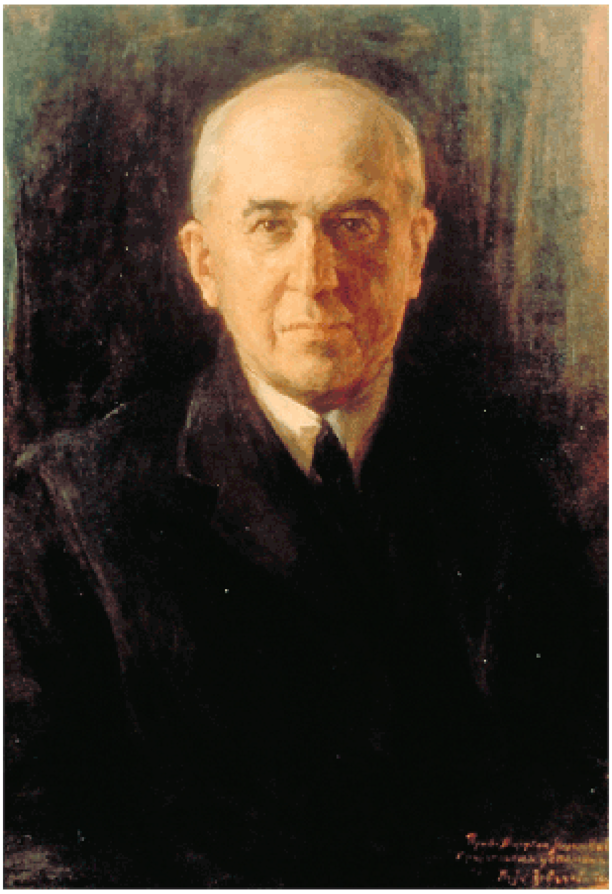

Figure 5. Milutin Milanković's portrait (Paja Jovanović, 1943)

Milutin Milanković is the most cited Serbian scientist.

REFERENCES

[1] Newton, I.: *Philosophiae Naturalis Principia Mathematica*, Nauka, Moscow, 1989, (in Russian).

[2] Milanković, M.: *Fundamentals of Celestial Mechanics*, Naučna knjiga, Belgrade, 1947, (in Serbian).

[3] http://sr.wikipedia.org/wiki/Милутин_Миланковић

МИЛАНКОВИЋЕВА АНАЛИЗА ЊУТНОВОГ ЗАКОНА СИЛЕ ОПШТЕ ГРАВИТАЦИЈЕ

Лазар Русов

У првом делу наведен је Њутнов поступак у дефинисању закона промене Њутнове силе опште гравитације. У наставку је приказан Миланковићев аналитички метод у одређивању закона промене дејства силе опште гравитације.