Two-Dimensional Transient Problem for a Thick Disc with Internal Heat Source

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The problem treated here is to determine thermo-elastic stress in a thick disc due to interior heat generation within the solid, under thermal boundary condition, which is subjected to arbitrary initial temperature on the upper and lower face at zero temperature, and the fixed circular edges with additional sectional heat supply. The governing heat conduction equation has been solved by using integral transformation techniques. The results are obtained in series form in terms of Bessel's functions. Numerical calculations are carried out for a particular case of a disc made of Aluminum metal and the results are depicted in figures.

Keywords: *transient response, disc, temperature distribution, thermal stress, integral transform.*

1. INTRODUCTION

As a result of the increased usage of industrial and construction materials the interest in isotropic thermal stress problems has grown considerably. However, there are only a few studies concerned with the two-dimensional steady-state thermal stress. Nowacki [1] has determined steady-state thermal stresses in a thick circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Wankhede [2] has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature. However, there are not many investigations on transient state. Roy Choudhuri [3] has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of circular upper face with lower face at zero temperature and the fixed circular edge thermally insulated. In a recent work, some problems have been solved by Noda et al. [4] and Deshmukh et al. [5]. In all aforementioned investigations an axisymmetrically heated plate has been considered. Recently, Nasser [6,7] proposed the concept of heat sources in generalized thermoelasticity and applied to a thick plate problem. They have not, however, considered any thermoelastic problem with boundary conditions of radiation type, in which sources are generated according to the linear function of the temperatures, which satisfies the time-dependent heat conduction equation. From the previous literatures regarding disc as considered, it was observed by the author that no analytical procedure has been established, considering internal heat sources generation within the body. The success of this novel research mainly lies with the new mathematical procedures with a much simpler approach for optimization for the design in terms of material usage and performance in engineering problem, particularly in the determination of thermoelastic behavior in disc engaged for pressure vessels, furnaces, etc.

This paper is concerned with the transient thermoelastic problem in a disc in which sources are generated according to the linear function of temperature, occupying the space $D = \{(x,y,z) \in R^3 : a \le (x^2 + y^2)^{1/2} \le b, -h \le z \le h\}$ where $r = (x^2 + y^2)^{1/2}$ with radiation type boundary conditions.

2. STATEMENT OF THE PROBLEM

In the first instance, we consider a disc in which sources are generated according to the linear function of temperature. The material of the disc is isotropic, homogenous and all properties are assumed to be constant. Heat conduction with internal heat source and the prescribed boundary conditions of the radiation type, the quasi-static thermal stresses are required to be determined. The equation for heat conduction is $\theta(r,z,t)$ the temperature, in cylindrical coordinates, is:

$$\kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{\partial^2\theta}{\partial z^2}\right]+\Theta\left(r,z,t,\theta\right)=\frac{\partial\theta}{\partial t} \tag{1}$$

where $\Theta(r,z,t,\theta)$ is the internal source function, and $\kappa = \lambda/\rho C$, λ being the thermal conductivity of the material, ρ is the density and C is the calorific capacity, assumed to be constant. For convenience, we consider the undergiven functions as the superposition of the simpler function [8]:

$$\Theta\left(r,z,t,\theta\right)=\Phi\left(r,z,t\right)+\psi\left(t\right)\theta\left(r,z,t\right) \tag{2}$$

and

$$T\left(r,z,t\right)=\theta\left(r,z,t\right)\exp\left[-\int_0^t\psi\left(\zeta\right)\mathrm{d}\zeta\right],$$

$$\chi\left(r,z,t\right)=\Phi\left(r,z,t\right)\exp\left[-\int_0^t\psi\left(\zeta\right)\mathrm{d}\zeta\right] \tag{3}$$

or for the sake of brevity, we consider

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$$\chi(r,z,t)=\frac{\delta(r-r_0)\delta(z-z_0)}{2\pi r_0}\exp(-\omega t),$$
$$a\le r_0\le b\,,\ -h\le z_0\le h\,,\ \omega>0\,. \tag{3a}$$

Substituting (2) and (3) to the heat conduction equation (1), one obtains

$$\kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]+\chi(r,z,t)=\frac{\partial T}{\partial t} \tag{4}$$

where κ is the thermal diffusivity of the material of the disc (which is assumed to be constant), subject to the initial and boundary conditions

$$M_t(T,1,0,0)=T_0$$
$$\text{for all } a\le r\le b\,,\ -h\le z\le h \tag{5}$$

$$M_r(T,1,k_1,a)=0\,,$$
$$M_r(T,1,k_2,b)=0$$
$$\text{for all } -h\le z\le h\,,\ t>0 \tag{6}$$

$$M_z(T,1,k_3,h)=\exp(-\omega t)\delta(r-r_0)\,,$$
$$M_z(T,1,k_4,-h)=\exp(-\omega t)\delta(r-r_0)$$
$$\text{for all } 0\le r\le b\,,\ t>0\,. \tag{7}$$

The most general expression for these conditions can be given by

$$M_\vartheta\left(f,\bar{k},\bar{\bar{k}},\$\right)=\left(\bar{k}\,f+\bar{\bar{k}}\,\hat{f}\right)_{\vartheta=\$} \tag{7a}$$

where the prime (^) denotes differentiation with respect to ϑ; $\delta(r-r_0)$ is the Dirac Delta function having $a\le r_0\le b$; $\omega>0$ is a constant; $\exp(-\omega t)\delta(r-r_0)$ is the additional sectional heat supply available on its surface at $z=\pm h$; T_0 is the reference temperature; $\bar{k}$ (= 1) and $\bar{\bar{k}}$ (= k_i, i = 1,2,3,4) are radiation coefficients of the disc, respectively.

The Navier's equations without the body forces for axisymmetric two-dimensional thermoelastic problem can be expressed as [4]

$$\nabla^2 u_r-\frac{u_r}{r}+\frac{1}{1-2\upsilon}\frac{\partial e}{\partial r}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial r}=0\,,$$
$$\nabla^2 u_z-\frac{1}{1-2\upsilon}\frac{\partial e}{\partial z}-\frac{2(1+\upsilon)}{1-2\upsilon}\alpha_t\frac{\partial\theta}{\partial z}=0 \tag{8}$$

where u_r and u_z are the displacement components in the radial and axial directions, respectively and the dilatation e as

$$e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\,. \tag{8a}$$

The displacement function in the cylindrical coordinate system are represented by the Goodier's thermoelastic displacement potential $\phi(r,z,t)$ and Love's function L as [5]

$$u_r=\frac{\partial\phi}{\partial r}-\frac{\partial^2 L}{\partial r\partial z}\,, \tag{9}$$

$$u_z=\frac{\partial\phi}{\partial z}+2(1-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial^2 z} \tag{10}$$

in which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2\phi=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\theta \tag{11}$$

and the Love's function L must satisfy the equation

$$\nabla^2\left(\nabla^2 L\right)=0 \tag{12}$$

where

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}\,. \tag{12a}$$

The components of the stresses are represented by the use of the potential ϕ and Love's function L as

$$\sigma_{rr}=2G\left\{\left(\frac{\partial^2\phi}{\partial r^2}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left(\upsilon\nabla^2 L-\frac{\partial^2 L}{\partial r^2}\right)\right\}, \tag{13}$$

$$\sigma_{\theta\theta}=2G\left\{\left(\frac{1}{r}\frac{\partial\phi}{\partial r}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left(\upsilon\nabla^2 L-\frac{1}{r}\frac{\partial L}{\partial r}\right)\right\}, \tag{14}$$

$$\sigma_{zz}=2G\left\{\left(\frac{\partial^2\phi}{\partial z^2}-\nabla^2\phi\right)+\frac{\partial}{\partial z}\left((2-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial z^2}\right)\right\}, \tag{15}$$

and

$$\sigma_{rz}=2G\left\{\frac{\partial^2\phi}{\partial r\partial z}+\frac{\partial}{\partial r}\left((1-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial z^2}\right)\right\} \tag{16}$$

where G and υ are the shear modulus and Poisson's ratio respectively.

The boundary condition on the traction free surface stress functions is

$$\sigma_{zz}\big|_{z=\pm h}=\sigma_{rz}\big|_{z=\pm h}=0\,. \tag{17}$$

The equations (1) – (17) constitute the mathematical formulation of the problem under consideration.

3. SOLUTION OF THE PROBLEM

3.1 Transient heat conduction analysis

In order to solve fundamental differential equation (4) under the boundary condition (6), we firstly introduce the integral transform [9] of order n over the variable r. Let n be the parameter of the transform, then the integral transform and its inversion theorem are written as

$$\bar{g}(n)=\int_a^b r g(r)S_p(k_1,k_2,\mu_n r)\mathrm{d}r\,,$$
$$g(r)=\sum_{n=1}^{\infty}\left(\bar{g}_p(n)/C_n\right)S_p(k_1,k_2,\mu_n r) \tag{18}$$

where $\overline{g}_p(n)$ is the transform of $g(r)$ with respect to nucleus $S_p(k_1,k_2,\mu_n r)$ and the eigenvalues μ_n are the positive roots of the characteristic equation given as

$$J_0(k_1,\mu a)Y_0(k_2,\mu b)-J_0(k_2,\mu b)Y_0(k_1,\mu a)=0\,. \quad (18a)$$

The kernel function $S_0(k_1,k_2,\mu_n r)$ in the interval $a \le r \le b$ can be defined as

$$S_0(k_1,k_2,\mu_n r)=J_0(\mu_n r)\left[Y_0(k_1,\mu_n a)+Y_0(k_2,\mu_n b)\right]- -Y_0(\mu_n r)\left[J_0(k_1,\mu_n a)+J_0(k_2,\mu_n b)\right] \quad (18b)$$

with

$$J_0(k_i,\mu r)=J_0(\mu r)+k_i\mu J_0'(\mu r)\,,$$
$$Y_0(k_i,\mu r)=Y_0(\mu r)+k_i\mu Y_0'(\mu r) \quad (18c)$$

for $i = 1,2$ and

$$C_n=\int_a^b r\left[S_0(k_1,k_2,\mu_n b)\right]^2 \mathrm{d}r \quad (18d)$$

in which $J_0(\mu r)$ and $Y_0(\mu r)$ are Bessel functions of first and second kind of order $p = 0$ respectively.

Applying the transform defined in (18) to the (3) – (5) and (7), and taking into account (6), one obtains

$$\kappa\left[-\mu_n^2\overline{T}(n,z,t)+\frac{\partial^2\overline{T}(n,z,t)}{\partial z^2}\right]+\overline{\chi}(n,z,t)= =\frac{\partial\overline{T}(n,z,t)}{\partial t}\,, \quad (19)$$

$$M_t\left(\overline{T},1,0,0\right)=\overline{T}_0\,, \quad (20)$$

$$M_z(T,1,k_3,h)=\exp(-\omega t)r_0S_0(k_1,k_2,\mu_n r_0)\,,$$
$$M_z(T,1,k_4,-h)=\exp(-\omega t)r_0S_0(k_1,k_2,\mu_n r_0)\,, \quad (21)$$

$$\overline{\chi}(n,z,t)= =r_0S_0(k_1,k_2,\mu_n r_0)\delta(z-z_0)\exp(-\omega t) \quad (22)$$

where $\overline{T}$ is the transformed function of T and n is the transform parameter.

We introduce another integral transform [8] that responds to the boundary conditions of type (7)

$$\overline{f}(m,t)=\int_{-h}^{h} f(z,t)P_m(z)\mathrm{d}z\,,$$
$$f(z,t)=\sum_{m=1}^{\infty}\frac{\overline{f}(m,t)}{\lambda_m}P_m(z)\,. \quad (23)$$

The symbol ($\overline{\ }$) means a function in the transformed domain, and the nucleus is given by the orthogonal functions in the interval $-h \le z \le h$ as

$$P_m(z)=Q_m\cos(a_m z)-W_m\sin(a_m z) \quad (23a)$$

where

$$Q_m=a_m(k_3+k_4)\cos(a_m h)\,,$$
$$W_m=2\cos(a_m h)+(k_3-k_4)a_m\sin(a_m h)\,,$$

$$\lambda_m=\int_{-h}^{h}P_m^2(z)\mathrm{d}z=$$
$$=h\left[Q_m^2+W_m^2\right]+\frac{\sin(2a_m h)}{2a_m}\left[Q_m^2-W_m^2\right] \quad (23b)$$

and the eigenvalues a_m are the positive roots of the characteristic equation with

$$\left[k_3 a\cos(ah)+\sin(ah)\right]\left[\cos(ah)+k_4 a\sin(ah)\right]=$$
$$=\left[k_4 a\cos(ah)-\sin(ah)\right]\left[\cos(ah)-k_3 a\sin(ah)\right]. \quad (23c)$$

Further applying the transform defined in (23) to the (19), (20) and (22), and using (21) one obtains

$$-\mu_n^2\,\overline{T}^*(n,m,t)+$$
$$+\left\{\frac{P_m(h)}{k_3}-\frac{P_m(-h)}{k_4}\right\}r_0S_0(k_1,k_2,\mu_n r_0)\exp(-\omega t)\,,$$
$$-a_m^2\,\overline{T}^*(n,m,t)+\frac{\overline{\chi}^*(n,m,t)}{\kappa}=\frac{1}{\kappa}\frac{\mathrm{d}\overline{T}^*(n,m,t)}{\mathrm{d}t}\,, \quad (24)$$

$$M_t\left(\overline{T}^*,1,0,0\right)=\overline{T}_0^*\,, \quad (25)$$

$$\overline{\chi}^*(n,m,t)=\frac{S_0(k_1,k_2,\mu_n r_0)P_m(z_0)\exp(-\omega t)}{2\pi} \quad (26)$$

where $\overline{T}^*$ is the transformed function of $\overline{T}$ and m is the transform parameter.

After performing some calculations on (24) and using (26), the reduction is made to linear first order differential equation as

$$\frac{\mathrm{d}\overline{T}^*}{\mathrm{d}t}+\kappa\Lambda_{n,m}\overline{T}^*=H(\mu_n,a_m) \quad (27)$$

where

$$\Lambda_{n,m}=\mu_n^2+a_m^2 \quad (27a)$$

and

$$H(\mu_n,a_m)=\left\{\frac{P_m(h)\kappa}{k_3}-\frac{P_m(-h)\kappa}{k_4}+\frac{P_m(z_0)}{2\pi r_0}\right\}\cdot$$
$$\cdot r_0S_0(k_1,k_2,\mu_n r_0)\exp(-\omega t)\,. \quad (27b)$$

The general solution of (27) is a function

$$\overline{T}^*(n,m,t)\exp\left(\kappa\Lambda_{n,m}t\right)=$$
$$=\frac{H(\mu_n,\alpha_m)}{\kappa\Lambda_{n,m}-\omega}\exp\left(-\kappa\Lambda_{n,m}t\right)+C\,. \quad (28)$$

Using (25) in (28), we obtain the values of arbitrary constants C. Substituting these values in (28) one obtains the transformed temperature solution as

$$\overline{T}^*(n,m,t)=\frac{H(\mu_n,a_m)}{\kappa\Lambda_{n,m}-\omega}\exp(-\omega t)+$$
$$+\left[\overline{T}_0^*-\frac{H(\mu_n,a_m)}{\kappa\Lambda_{n,m}-\omega}\right]\exp\left(-\kappa\Lambda_{n,m}t\right). \quad (29)$$

Applying inversion theorems of transformation rules defined in (23) to the (29), there results

$$\bar{T}(n,z,t)=\sum_{m=1}^{\infty}\frac{1}{\lambda_m}\left[\wp_{n,m}\exp(-\omega t)+\left(\bar{T}_0^*-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]P_m(z) \tag{30}$$

and then accomplishing inversion theorems of transformation rules defined in (18) on (30), the temperature solution is shown as follows:

$$T(r,z,t)=\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{1}{\lambda_m}\left[\wp_{n,m}\exp(-\omega t)+\left(\bar{T}_0^*-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}P_m(z)S_0(k_1,k_2,\mu_n r) \tag{31}$$

where

$$\wp_{n,m}=\frac{H(\mu_n,a_m)}{\kappa\Lambda_{n,m}-\omega}. \tag{31a}$$

Taking into account the first equation of (3), the temperature distribution is finally represented by

$$\theta(r,z,t)=\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{1}{\lambda_m}\left[\wp_{n,m}\exp(-\omega t)+\left(\bar{T}_0^*-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}P_m(z)S_0(k_1,k_2,\mu_n r)\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right]. \tag{32}$$

The function given in (32) represents the temperature at every instant and at all points of disc of finite height when there are conditions of radiation type.

3.2 Thermoelastic solution

Referring to the fundamental equation (1) and its solution (32) for the heat conduction problem, the solution for the displacement function is represented by the Goodier's thermoelastic displacement potential ϕ governed by (11) and represented by

$$\phi(r,z,t)=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m\Lambda_{n,m}}\left[\wp_{n,m}\exp(-\omega t)+\left(\bar{T}_0^*-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]P_m(z)\right\}\cdot$$
$$\cdot S_0(k_1,k_2,\mu_n r)\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right]. \tag{33}$$

Similarly, the solution for Love's function L is assumed so as to satisfy the governed condition of (12) as

$$L(r,z,t)=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m\Lambda_{n,m}}\left[\wp_{n,m}\exp(-\omega t)+\left(\bar{T}_0^*-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot$$
$$\cdot\left[B_{nm}\sinh(\mu_n z)+C_{nm}z\cosh(\mu_n z)\right]S_0(k_1,k_2,\mu_n r)\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right]. \tag{34}$$

In this manner two displacement functions in the cylindrical coordinate system ϕ and Love's function L are formulated. Now, in order to obtain the displacement components, substituting the values of thermoelastic displacement potential ϕ and Love's function L in (9) and (10), one obtains

$$u_r=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m\Lambda_{n,m}}\left[\wp_{n,m}\exp(-\omega t)+\left(\bar{T}_0^*-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot$$
$$\cdot\left\{P_m(z)-\left[(B_{nm}\mu_n+C_{nm})\cosh(\mu_n z)+C_{nm}z\sinh(\mu_n z)\right]\right\}S_0'(k_1,k_2,\mu_n r)\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right], \tag{35}$$

$$u_z=\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m\Lambda_{n,m}}\left[\wp_{n,m}\exp(-\omega t)+\left(\bar{T}_0^*-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot$$
$$\cdot\left\{\left[-a_m\left(Q_m\sin(a_m z)+W_m\cos(a_m z)\right)-\mu_n^2(-1+2\upsilon)\left(B_{nm}\sinh(\mu_n z)+C_{nm}z\cosh(\mu_n z)\right)-2(-1+2\upsilon)C_{nm}\sinh(\mu_n z)\mu_n\right]\cdot\right.$$
$$\cdot S_0(k_1,k_2,\mu_n r)+\mu_n\left(2(1-\upsilon)\right)\left[B_{nm}\sinh(\mu_n z)+C_{nm}z\cosh(\mu_n z)\right]\cdot$$
$$\left.\cdot\left[\mu_n S_0''(k_1,k_2,\mu_n r)+r^{-1}S_0'(k_1,k_2,\mu_n r)\right]\right\}\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right]. \tag{36}$$

Thus, making use of the two displacement components, the dilation can be obtained. Then, the stress components can be evaluated by substituting the values of thermoelastic displacement potential ϕ from (33) and Love's function L from (34) in (13) – (15) and (16), one obtains

$$\sigma_{rr} = 2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m \Lambda_{nm}}\left[\wp_{n,m}\exp(-\omega t)+\left(\overline{T}_0^{*}-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot$$
$$\cdot\Big\{-P_m(z)\left[r^{-1}S_0'(k_1,k_2,\mu_n r)-a_m^2 S_0(k_1,k_2,\mu_n r)\right]+\mu_n^2(\upsilon-1)\left[\mu_n B_{nm}\cosh(\mu_n z)+C_{nm}\left(z\sinh(\mu_n z)+\cosh(\mu_n z)\right)\right]\cdot$$
$$\cdot S_0''(k_1,k_2,\mu_n r)+\mu_n\upsilon\left[B_{nm}\cosh(\mu_n z)+C_{nm}\left(z\sinh(\mu_n z)+\cosh(\mu_n z)\right)\right]\cdot$$
$$\cdot\left[r^{-1}S_0'(k_1,k_2,\mu_n r)+\mu_n S_0(k_1,k_2,\mu_n r)\right]+2\upsilon C_{nm}\mu_n^2\cosh(\mu_n z)S_0(k_1,k_2,\mu_n r)\Big\}\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right], \tag{37}$$

$$\sigma_{\theta\theta} = 2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m \Lambda_{nm}}\left[\wp_{n,m}\exp(-\omega t)+\left(\overline{T}_0^{*}-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot$$
$$\cdot\Big\{-P_m(z)\left[\mu_m^2 S_0''(k_1,k_2,\mu_n r)-a_m^2 S_0(k_1,k_2,\mu_n r)\right]+\mu_n(\upsilon-1)r^{-1}\left[(\mu_n B_{nm}+C_{nm})\cosh(\mu_n z)+\mu_n C_{nm}z\sinh(\mu_n z)\right]\cdot$$
$$\cdot S_0(k_1,k_2,\mu_n r)+\mu_n^2\upsilon\left[(\mu_n B_{nm}+C_{nm})\cosh(\mu_n z)+\mu_n C_{nm}z\sinh(\mu_n z)\right]\cdot$$
$$\cdot\left[S_0''(k_1,k_2,\mu_n r)+S_0(k_1,k_2,\mu_n r)\right]+2\upsilon C_{nm}\mu_n^2\cosh(\mu_n z)S_0(k_1,k_2,\mu_n r)\Big\}\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right], \tag{38}$$

$$\sigma_{zz} = 2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m \Lambda_{nm}}\left[\wp_{n,m}\exp(-\omega t)+\left(\overline{T}_0^{*}-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot$$
$$\cdot\Big\{-\mu_n P_m(z)\left[\mu_n S_0''(k_1,k_2,\mu_n r)+r^{-1}S_0(k_1,k_2,\mu_n r)\right]+\mu_n^2\left[B_{nm}\cosh(\mu_n z)+C_{nm}z\sinh(\mu_n z)\right]\cdot$$
$$\cdot(2-\upsilon)\left[\mu_n S_0''(k_1,k_2,\mu_n r)+r^{-1}S_0(k_1,k_2,\mu_n r)\right]+(1-\upsilon)\mu_n S_0(k_1,k_2,\mu_n r)+\mu_n C_{nm}\cosh(\mu_n z)\cdot$$
$$\cdot\left[(2-\upsilon)\left[\mu_n S_0''(k_1,k_2,\mu_n r)+r^{-1}S_0(k_1,k_2,\mu_n r)\right]+(1-\upsilon)\mu_n^2 S_0(k_1,k_2,\mu_n r)\right]\Big\}\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right], \tag{39}$$

$$\sigma_{rz} = 2G\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-1}{\lambda_m \Lambda_{nm}}\left[\wp_{n,m}\exp(-\omega t)+\left(\overline{T}_0^{*}-\wp_{n,m}\right)\exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot$$
$$\cdot\Big\{\left[-\mu_n a_m\left(Q_m\sin(a_m z)+W_m\cos(a_m z)\right)\right]S_0(k_1,k_2,\mu_n r)+$$
$$+\left[B_{nm}\sinh(\mu_n z)+C_{nm}z\cosh(\mu_n z)\mu_n^2\left[\upsilon\mu_n+(1-\upsilon)r^{-1}\right]-2\upsilon\mu_n^2 C_{nm}\sinh(\mu_n z)\right]S_0'(k_1,k_2,\mu_n r)+$$
$$+(1-\upsilon)\left[B_{nm}\sinh(\mu_n z)+C_{nm}z\cosh(\mu_n z)\right]\left[\mu_n^3 S_0'''(k_1,k_2,\mu_n r)-r^{-1}S_0(k_1,k_2,\mu_n r)\right]\Big\}\exp\left[\int_0^t\psi(\zeta)\,\mathrm{d}\zeta\right]. \tag{40}$$

3.3 Determination of unknown arbitrary function B_{nm} and C_{nm}

Applying boundary condition (17) to the (37) and (40) one obtains

$$B_{nm} = \frac{P_m(h)\overline{X}\left[h\cos(\mu_n h)\overline{Y}-\overline{Z}\right]-a_m\overline{f}\mu_n S_0'(k_1,k_2,\mu_n r)\overline{g}\left[(2-\upsilon)\overline{X}+(1-\upsilon)\mu_n S_0(k_1,k_2,\mu_n r)\right]}{\left[(2-\upsilon)\overline{X}+(1-\upsilon)\mu_n S_0(k_1,k_2,\mu_n r)\right]\left\{(h\mu_n)\cos(\mu_n h)\left[h\cos(\mu_n h)\overline{Y}-\overline{Z}\right]-\overline{g}\sinh(\mu_n h)\overline{Y}\right\}} \tag{41}$$

and

$$C_{nm} = \frac{P_m(h)\overline{X}\left[\sinh(\mu_n h)\overline{Y}\right]-a_m\overline{f}\mu_n S_0'(k_1,k_2,\mu_n r)\cosh(\mu_n h)\left[(2-\upsilon)\overline{X}+(1-\upsilon)\mu_n S_0(k_1,k_2,\mu_n r)\right]}{\left[(2-\upsilon)\overline{X}+(1-\upsilon)\mu_n S_0(k_1,k_2,\mu_n r)\right]\left\{\overline{g}\sinh(\mu_n h)\overline{Y}-(h\mu_n)\cos(\mu_n h)\left[h\cos(\mu_n h)\overline{Y}-\overline{Z}\right]\right\}} \tag{42}$$

where

$$\bar{X} = S_0''\left(k_1,k_2,\mu_n r\right) + r^{-1} S_0\left(k_1,k_2,\mu_n r\right),$$
$$\bar{Y} = \mu_n^2\left(\upsilon\mu_n + r^{-1}\left(1-\upsilon\right)\right)S_0'\left(k_1,k_2,\mu_n r\right) + \left(1-\upsilon\right)\left(\mu_n^3 S_0'''\left(k_1,k_2,\mu_n r\right) - r^{-2} S_0\left(k_1,k_2,\mu_n r\right)\right),$$
$$\bar{Z} = 2\upsilon\mu_n^2 \sinh\left(\mu_n h\right) S_0'\left(k_1,k_2,\mu_n r\right),$$
$$\bar{f} = W_m \cos\left(a_m h\right) + Q_m \sin\left(a_m h\right),$$
$$\bar{g} = \cosh\left(\mu_n h\right) + \left(\mu_n h\right)\sinh\left(\mu_n h\right). \tag{42a}$$

4. SPECIAL CASE

Set

$$\psi\left(\zeta\right) = \delta\left(t - t_0\right),\ 0 < t_0 < t, \tag{43}$$

$$\bar{T}_0^* = 0. \tag{44}$$

Substituting (43) and (44) into (32) and (37) – (40), one obtains the expressions for the temperature and stresses respectively as follows:

$$\theta\left(r,z,t\right) = e\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{\wp_{n,m}}{\lambda_m}\left[\exp\left(-\omega t\right) - \exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}P_m\left(z\right)S_0\left(k_1,k_2,\mu_n r\right), \tag{45}$$

$$\begin{aligned}
\sigma_{rr} &= 2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-\wp_{n,m}}{\lambda_m\Lambda_{nm}}\left[\exp\left(-\omega t\right) - \exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot \\
&\cdot\left\{-P_m\left(z\right)\left[r^{-1}S_0'\left(k_1,k_2,\mu_n r\right) - a_m^2 S_0\left(k_1,k_2,\mu_n r\right)\right] + \mu_n^2\left(\upsilon-1\right)\left[\mu_n B_{nm}\cosh\left(\mu_n z\right) + C_{nm}\left(z\sinh\left(\mu_n z\right) + \cosh\left(\mu_n z\right)\right)\right]\cdot\right. \\
&\cdot S_0''\left(k_1,k_2,\mu_n r\right) + \mu_n\upsilon\left[B_{nm}\cosh\left(\mu_n z\right) + C_{nm}\left(z\sinh\left(\mu_n z\right) + \cosh\left(\mu_n z\right)\right)\right]\cdot \\
&\left.\cdot\left[r^{-1}S_0'\left(k_1,k_2,\mu_n r\right) + \mu_n S_0\left(k_1,k_2,\mu_n r\right)\right] + 2\upsilon C_{nm}\mu_n^2\cosh\left(\mu_n z\right)S_0\left(k_1,k_2,\mu_n r\right)\right\},
\end{aligned} \tag{46}$$

$$\begin{aligned}
\sigma_{\theta\theta} &= 2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-\wp_{n,m}}{\lambda_m\Lambda_{nm}}\left[\exp\left(-\omega t\right) - \exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot \\
&\cdot\left\{-P_m\left(z\right)\left[\mu_m^2 S_0''\left(k_1,k_2,\mu_n r\right) - a_m^2 S_0\left(k_1,k_2,\mu_n r\right)\right] + \mu_n\left(\upsilon-1\right)r^{-1}\left[\left(\mu_n B_{nm} + C_{nm}\right)\cosh\left(\mu_n z\right) + \mu_n C_{nm} z\sinh\left(\mu_n z\right)\right]\cdot\right. \\
&\cdot S_0\left(k_1,k_2,\mu_n r\right) + \mu_n^2\upsilon\left[\left(\mu_n B_{nm} + C_{nm}\right)\cosh\left(\mu_n z\right) + \mu_n C_{nm} z\sinh\left(\mu_n z\right)\right]\cdot \\
&\left.\cdot\left[S_0''\left(k_1,k_2,\mu_n r\right) + S_0\left(k_1,k_2,\mu_n r\right)\right] + 2\upsilon C_{nm}\mu_n^2\cosh\left(\mu_n z\right)S_0\left(k_1,k_2,\mu_n r\right)\right\},
\end{aligned} \tag{47}$$

$$\begin{aligned}
\sigma_{zz} &= 2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-\wp_{n,m}}{\lambda_m\Lambda_{nm}}\left[\exp\left(-\omega t\right) - \exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot \\
&\cdot\left\{-\mu_n P_m\left(z\right)\left[\mu_n S_0''\left(k_1,k_2,\mu_n r\right) + r^{-1}S_0\left(k_1,k_2,\mu_n r\right)\right] + \mu_n^2\left[B_{nm}\cosh\left(\mu_n z\right) + C_{nm} z\sinh\left(\mu_n z\right)\right]\cdot\right. \\
&\cdot\left(2-\upsilon\right)\left[\mu_n S_0''\left(k_1,k_2,\mu_n r\right) + r^{-1}S_0\left(k_1,k_2,\mu_n r\right)\right] + \left(1-\upsilon\right)\mu_n S_0\left(k_1,k_2,\mu_n r\right) + \mu_n C_{nm}\cosh\left(\mu_n z\right)\cdot \\
&\left.\cdot\left[\left(2-\upsilon\right)\left[\mu_n S_0''\left(k_1,k_2,\mu_n r\right) + r^{-1}S_0\left(k_1,k_2,\mu_n r\right)\right] + \left(1-\upsilon\right)\mu_n^2 S_0\left(k_1,k_2,\mu_n r\right)\right]\right\},
\end{aligned} \tag{48}$$

$$\begin{aligned}
\sigma_{rz} &= 2eG\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_{\mathrm{t}}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}\frac{-\wp_{n,m}}{\lambda_m\Lambda_{nm}}\left[\exp\left(-\omega t\right) - \exp\left(-\kappa\Lambda_{n,m}t\right)\right]\right\}\cdot \\
&\cdot\left\{\left[-\mu_n a_m\left(Q_m\sin\left(a_m z\right) + W_m\cos\left(a_m z\right)\right)\right]S_0\left(k_1,k_2,\mu_n r\right) + \right. \\
&+\left[B_{nm}\sinh\left(\mu_n z\right) + C_{nm} z\cosh\left(\mu_n z\right)\mu_n^2\left[\upsilon\mu_n + \left(1-\upsilon\right)r^{-1}\right] - 2\upsilon\mu_n^2 C_{nm}\sinh\left(\mu_n z\right)\right]S_0'\left(k_1,k_2,\mu_n r\right) + \\
&\left.+\left(1-\upsilon\right)\left[B_{nm}\sinh\left(\mu_n z\right) + C_{nm} z\cosh\left(\mu_n z\right)\right]\left[\mu_n^3 S_0'''\left(k_1,k_2,\mu_n r\right) - r^{-1}S_0\left(k_1,k_2,\mu_n r\right)\right]\right\}.
\end{aligned} \tag{49}$$

5. NUMERICAL RESULTS, DISCUSSION AND REMARKS

To interpret the numerical computations, we consider material properties of aluminum, which can be commonly used in both, wrought and cast forms. The low density of aluminum results in its extensive use in the aerospace industry, and in other transportation fields. Its resistance to corrosion leads to its use in food and chemical handling (cookware, pressure vessels, etc.) and to architectural uses.

Table 1. Material properties and parameters used in this study (property values are nominal) [10]

Material propertie / parameter	Value
Modulus of Elasticity, E [N/cm^2]	$6.9 \cdot 10^6$
Shear modulus, G [N/cm^2]	$2.7 \cdot 10^6$
Poisson ratio, υ	0.281
Thermal expansion coefficient, α_t [1/°C]	$25.5 \cdot 10^{-6}$
Thermal diffusivity, κ [cm^2/s]	0.86
Thermal conductivity, λ [W/m °C] (cal s^{-1}/cm °C)	200.96 (0.48)
Inner radius, a [cm]	1
Outer radius, b [cm]	4
Thickness, h [cm]	2

In the foregoing analyses are performed by setting the radiation coefficients constants, $k_i = 0.86$ ($i = 1,3$) and $k_i = 1$ ($i = 2,4$), so as to obtain considerable mathematical simplicities. The other parameters considered are $r_0 = 2.5$, $z_0 = 1$ and $\omega = 1$.

The derived numerical results from (45) – (49) have been illustrated graphically (Figs. 1 – 5) for both: with internal heat source, as well as without internal heat source, with available additional sectional heat on its flat surface at $z = 1$.

Figure 1 shows the temperature distribution along the radial and thickness direction of the disc at $t = 0.25$. It is observed that due to the thickness of the disc, a steep increase in temperature was found at the beginning of the transient period. As expected, temperature drop becomes more and more gradually along thickness direction.

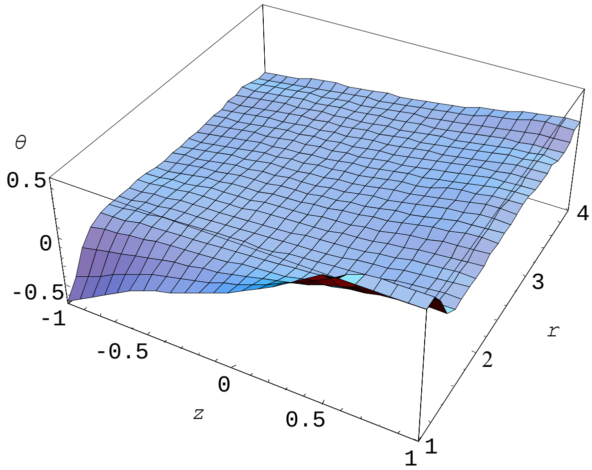

Figure 1. Temperature distribution *θ* along *r* and *z* direction for *t* = 0.25 with internal heat source

Figure 2 shows the radial stress distribution σ_{rr} along the radial and thickness direction of the disc at $t = 0.25$. From the figure, the location of points of minimum stress occurs at the end points through-the-thickness direction, while the thermal stress response is maximal at the interior and so that outer edges tend to expand more than the inner surface leading inner part being under tensile stress.

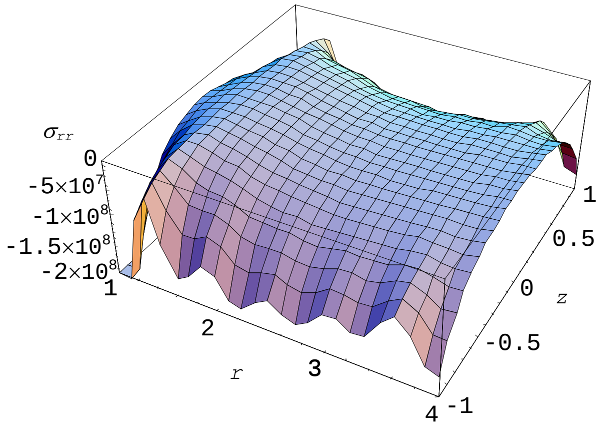

Figure 2. Radial stress distribution σ_{rr} for varying along *r* axis and *z* axis for *t* = 0.25 with internal heat source

Figure 3 shows the tangential stress distribution $\sigma_{\theta\theta}$ along the radial and thickness direction of the disc at $t = 0.25$. The tangential stress follows a sinusoidal nature with high crest and troughs at both ends i.e. $r = 1$ and $r = 4$.

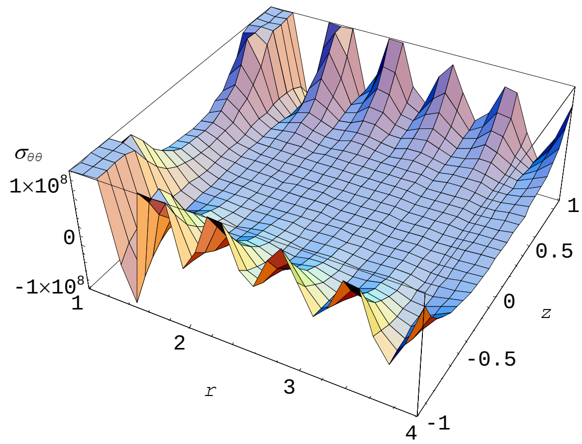

Figure 3. Tangential stress distribution $\sigma_{\theta\theta}$ for varying along *r* axis and *z* axis for *t* = 0.25 with internal heat source

Figure 4 shows the axial stress distribution σ_{zz}, which is similar in nature, but small in magnitude as compared to radial stress component.

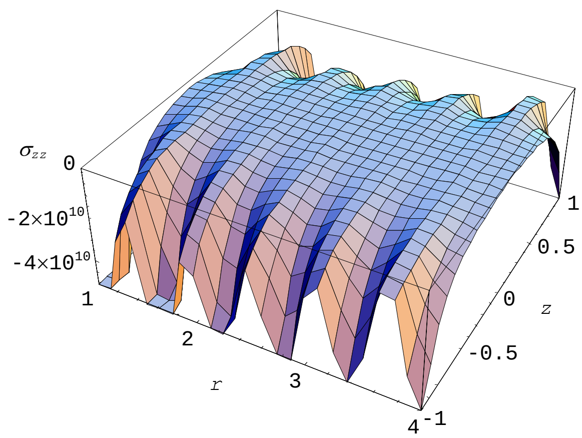

Figure 4. Axial stress distribution σ_{zz} for varying along *r* axis and *z* axis for *t* = 0.25 with internal heat source

Figure 5 shows the shear stress distribution σ_{rz} along the radial and thickness direction of the disc at $t = 0.25$. Shear stress also follows more sine waveform with high pecks and troughs along the radial direction at $r = 1$ and $r = 4$, but minimum at the center part along thickness direction.

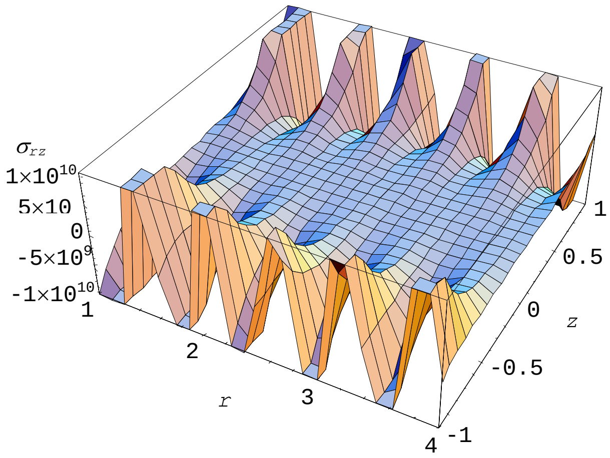

Figure 5. Shear stress distribution σ_{rz} for varying along *r* axis and *z* axis for *t* = 0.25 with internal heat source

In order for the solution to be meaningful the series expressed in (32) should converge for all $a \le r \le b$ and $-h \le z \le h$. The temperature equation (32) can be expressed as

$$\theta(r,z,t) = e\sum_{n=1}^{M}\frac{1}{C_n}\cdot \cdot\left\{\sum_{m=1}^{M'}\frac{\wp_{n,m}}{\lambda_m}\left[\exp(-\omega t)-\exp(-\kappa\Lambda_{n,m}t)\right]\right\}\cdot \cdot P_m(z)S_0(k_1,k_2,\mu_n r). \qquad (50)$$

We impose conditions so that $\theta(r,z,t)$ converges in some generalized sense to $g(r,s)$ as $t \to 0$ in the transform domain. Taking into account the asymptotic behaviors of $P_m(z)$, μ_n, $S_0(k_1,k_2,\mu_n r)$ and C_n [8,9], it is observed that the series expansion for $\theta(r,z,t)$ will be theoretically convergent due to the bounded functions.

6. CONCLUSION

In this study, we treated the two-dimensional thermoelastic problem of a thick disc in which sources are generated according to the linear function of the temperature. We successfully established and obtained the temperature distribution, displacements and stress functions with additional sectional heat, $\exp(-\omega t)\delta(r - r_0)$ available at the edge $z = \pm h$ of the disc. Then, in order to examine the validity of two-dimensional thermoelastic boundary value problem, we analyzed a particular case with mathematical model for $\psi(\zeta) = \delta(t - t_0)$, and numerical calculations were carried out.

From the figures of stress functions it can be observed that radial and axial stresses develop tensile stress at the center of the disc, whereas its opposite happens at the outer circular boundary. Tangential and shear stresses develop compressive stress in axial direction, so it may be concluded that due arbitrary heat flux on the upper and lower surfaces of disc and internal heat generation expand in axial direction and bend concavely at the center. This expansion and bending is in the proportion of the thermal diffusivity of metal.

The results obtained here are more useful in engineering problems particularly in the determination of state of strain in thick disc. Also, any particular case of special interest can be derived by assigning suitable values to the parameters and function in the (37) – (40).

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ДВОДИМЕНЗИОНАЛНИ НЕСТАЦИОНАРНИ ПРОБЛЕМ ЗА ДИСК ВЕЛИКЕ ДЕБЉИНЕ ЗИДА СА УНУТРАШЊИМ ИЗВОРОМ ТОПЛОТЕ

Винод Варгезе Лалсинг Калса

Рад се бави проблемом дефинисања термопластичних напона који настају услед генерисања топлоте унутар диска велике дебљине зида у граничним топлотним условима. По горњој површини диск је изложен произвољној почетној температури, температура доње површине је нула, а попречни пресек фиксне кружне ивице изложен је додатном топлотном извору. Једначина топлотне проводљивости је решена техником интегралне трансформације. Резултати су добијени у облику реда помоћу Беселових функција. Нумерички прорачун је обављен за случај диска од алуминијума, а резултати су приказани на дијаграмима.