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A Review of Explicit Approximations of Colebrook's Equation

The most common explicit correlations for estimation of the friction factor in rough and smooth pipes are reviewed in this paper. Comparison of any friction factor equation with the Colebrook's equation was expressed trough the mean relative error, the maximal positive error, the maximal negative error, correlation ratio and standard deviation. The statistical comparison of different equations was also carried out using the "Model selection criterion" and "Akaike Information Criterion". It was found that the equation of Zigrang and Sylvester provides the most accurate value of friction factor, and that Haaland's equation is most suitable for hand calculations.

Keywords: *Colebrook's equation, friction factor, approximations, fluid mechanics, turbulent flow.*

1. INTRODUCTION

The determination of a single-phase friction factor of pipe is essential to a variety of industrial applications, such as single-phase flow systems, two-phase flow systems and supercritical flow systems. Typically, the method of choice for computing friction factor is the Colebrook's equation.

This equation is a combination of Prandtl-von Karman-Nikuradse smooth-pipe equation

$$\frac{1}{\sqrt{f}} = 2\log\left(Re\sqrt{f}\right) - 0.08 \tag{1}$$

and rough-pipe equation

$$\frac{1}{\sqrt{f}} = 1.14 - 2\log(\varepsilon) \tag{2}$$

where Re is the Reynolds number and ε is the relative pipe roughness. Equations (1) and (2) are known as PKN equations [1]. Using these equations and his own data gathered on commercial pipes, Colebrook [2] formed the following equation that covers the whole turbulent flow region

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{2.51}{Re\sqrt{f}} + \frac{\varepsilon}{3.7}\right) \tag{3}$$

Received: April 2011, Accepted: May 2011
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that became widely accepted design formula for turbulent friction in the range of $Re = 4000 - 10^8$ and $\varepsilon = 0 - 0.05$.

Due to its demonstrated applicability, the Colebrook's equation (3) has become the acceptable standard for calculation of the friction factor in turbulent regimes. It should be noted that Rouse [3] was the first to confirm Colebrook's equation (3) by his own measurements.

Equation (3) was plotted in 1944 by Moody [4] into what is now called the *Moody chart* for pipe friction (this chart is probably the most famous and useful figure in engineering fluid mechanics). The implicit form of (3) disables the quick estimation of friction factor in hand calculations. For this reason, a number of approximate explicit counterparts have been proposed in the last 60 years and a most recent and very good overview of these equations is given in [5-7].

The basic idea of these efforts is to introduce more parameters in equation, in order to obtain as good results as possible, or more precisely as close prediction as possible of a Colebrook's equation. These explicate equations were compared with Colebrook's equation as shown in Section 3.

2. EXPLICIT EQUATIONS FOR CALCULATION OF THE FRICTION FACTOR IN TURBULENT FLOW

The most widely used explicit approximations for the Colebrook's equation postulated since 1947 are synthesized in Table 1, in the order of publication. Additionally, this table contains the range of validity for each approximation cited as defined in the original paper.

Most of these approximations are typically valid over only a limited range of the Re and ε values encountered in practice.

Table 1. Various approximations of the Colebrook's equation

Eq. num.	Equation	Range	Ref.	Authors (year)
(4)	$f=0.0055\left[1+\left(20000\varepsilon+\frac{10^6}{Re}\right)^{1/3}\right]$	$Re = 4000 - 5\cdot 10^8$ $\varepsilon = 0 - 0.01$	[8]	Moody (1947)
(5)	$f=0.11\left(\frac{68}{Re}+\varepsilon\right)^{0.25}$	Not specified	[9]	Altshul (1952)
(6)	$f=0.53\varepsilon+0.094\varepsilon^{0.225}+88\varepsilon^{0.44}Re^{-1.62\varepsilon^{0.134}}$	$Re = 4000 - 5\cdot 10^7$ $\varepsilon = 0.00001 - 0.04$	[10]	Wood (1966)
(7)	$f=\left[-2\log\left(\frac{\varepsilon}{3.7}+\frac{7}{Re^{0.9}}\right)\right]^{-2}$	Not specified	[11]	Churchill (1973)
(8)	$f=\left[1.14-2\log\left(\varepsilon+\frac{21.25}{Re^{0.9}}\right)\right]^{-2}$	$Re = 5000 - 10^7$ $\varepsilon = 0.00004 - 0.05$	[12]	Jain (1976)
(9)	$f=\left[-2\log\left(\frac{\varepsilon}{3.7}+\frac{5.74}{Re^{0.9}}\right)\right]^{-2}$	$Re = 5000 - 10^8$ $\varepsilon = 0.000001 - 0.05$	[13]	Swamee, Jain (1976)
(10)	$f=\left\{-2\log\left[\frac{\varepsilon}{3.7065}-\frac{5.0452}{Re}\log\left(\frac{\varepsilon^{1.1098}}{2.8257}+\frac{5.8506}{Re^{0.8981}}\right)\right]\right\}^{-2}$	$Re = 4000 - 4\cdot 10^8$	[14]	Chen (1979)
(11)	$f=\left[-1.8\log\left(0.135\varepsilon+\frac{6.5}{Re}\right)\right]^{-2}$	$Re = 4000 - 4\cdot 10^8$ $\varepsilon = 0 - 0.05$	[15]	Round (1980)
(12)	$f=\left\{-2\log\left[\frac{\varepsilon}{3.7}-\frac{5.02}{Re}\log\left(\varepsilon-\frac{5.02}{Re}\log\left(\frac{\varepsilon}{3.7}+\frac{13}{Re}\right)\right)\right]\right\}^{-2}$	$Re = 4000 - 10^8$ $\varepsilon = 0.00004 - 0.05$	[16]	Zigrang, Sylvester (1982)
(13)	$f=\left\{-1.8\log\left[\left(\frac{\varepsilon}{3.7}\right)^{1.11}+\frac{6.9}{Re}\right]\right\}^{-2}$	$Re = 4000 - 10^8$ $\varepsilon = 0.000001 - 0.05$	[17]	Haaland (1983)
(14)	$A=0.11\left(\frac{68}{Re}+\varepsilon\right)^{0.25}$ If $A\geq 0.018$ then $f=A$ and if $A<0.018$ then $f=0.0028+0.85A$	$Re = 4000 - 10^8$ $\varepsilon = 0 - 0.05$	[18]	Tsal (1989)
(15)	$f=\left[-2\log\left(\frac{\varepsilon}{3.70}+\frac{95}{Re^{0.983}}-\frac{96.82}{Re}\right)\right]^{-2}$	$Re = 4000 - 10^8$ $\varepsilon = 0 - 0.05$	[19]	Manadilli (1997)
(16)	$f=\left\{-2\log\left[\frac{\varepsilon}{3.7065}-\frac{5.0272}{Re}\log\left(\frac{\varepsilon}{3.827}-\frac{4.567}{Re}\cdot\log\left(\left(\frac{\varepsilon}{7.79}\right)^{0.9924}+\left(\frac{5.3326}{208.82+Re}\right)^{0.9345}\right)\right)\right]\right\}^{-2}$	$Re = 3000 - 1.5\cdot 10^8$ $\varepsilon = 0 - 0.05$	[20]	Romeo, Royo, Monzon (2002)
(17)	$f=1.613\left[\ln\left(0.234\varepsilon^{1.1007}-\frac{60.525}{Re^{1.1105}}+\frac{56.291}{Re^{1.0712}}\right)\right]^{-2}$	$Re = 3000 - 10^8$ $\varepsilon = 0 - 0.05$	[21]	Fang (2011)
(18)	$\beta=\ln\frac{Re}{1.816\ln\left(\frac{1.1Re}{\ln(1+1.1Re)}\right)}$, $f=\left[-2\log\left(10^{-0.4343\beta}+\frac{\varepsilon}{3.71}\right)\right]^{-2}$	Not specified	[7]	Brkić (2011)
(19)	$\beta=\ln\frac{Re}{1.816\ln\left(\frac{1.1Re}{\ln(1+1.1Re)}\right)}$, $f=\left[-2\log\left(\frac{2.18\beta}{Re}+\frac{\varepsilon}{3.71}\right)\right]^{-2}$	Not specified	[7]	Brkić (2011)

3. STATISTICAL COMPARISON OF THE EQUATIONS

The statistical comparison of any friction factor equation with the Colebrook's equation can be done by the following procedure:

- Divide the range of possible *Re* and ε using appropriate pitch into *n* nodes.
- Calculate the friction factor $f_{\text{pred},i}$ by the individual approximate equation.
- Calculate friction factor value $f_{\text{C,i}}$ calculated with the Colebrook's equation ($f_{\text{C,i}}$ was calculated numerically within the range of error $\pm 10^{-8}$).
- Calculate the following parameters:
  - o the mean relative error

$$meanRE = \frac{1}{n}\sum_{i=1}^{n}\frac{\left|f_{\text{C},i} - f_{\text{pred},i}\right|}{f_{\text{C},i}} \tag{20}$$

  - o the maximal positive error

$$maxRE^{+} = \max\left(\frac{f_{\text{C},i} - f_{\text{pred},i}}{f_{\text{C},i}}\right) \tag{21}$$

  - o the maximal negative error

$$maxRE^{-} = \max\left(\frac{f_{\text{pred},i} - f_{\text{C},i}}{f_{\text{C},i}}\right) \tag{22}$$

  - o Θ, correlation ratio

$$\Theta = \sqrt{1 - \frac{\sum_{i=1}^{n}\left(f_{\text{C},i} - f_{\text{pred},i}\right)^2}{\sum_{i=1}^{n}\left(f_{\text{C},i} - f_{\text{C,av}}\right)^2}} \tag{23}$$

  - o Δ_{av}, standard deviation

$$\Delta_{\text{av}} = \sqrt{\frac{\sum_{i=1}^{n}\left(\frac{f_{\text{C},i} - f_{\text{pred},i}}{f_{\text{C},i}}\right)^2}{n}} \tag{24}$$

where $f_{\text{C,av}}$ is the average value of f_{C} for complete set of nods

$$f_{\text{C,av}} = \frac{\sum_{i=1}^{n} f_{\text{C},i}}{n}. \tag{25}$$

In this paper, we will use the range of $Re = 4000 - 10^8$ and $\varepsilon = 0 - 0.05$ and a net will be formed using linear scale with 10^6 nods.

Three ways were used to produce the number of nods, presented in Table 2.

Table 2. Three ways for forming the net with 10^6 nods

	Range	Nods	Linear step
I	$Re = 4000 - 10^8$	1000	99996
	$\varepsilon = 0 - 0.05$	1000	$50 \cdot 10^{-6}$
II	$Re = 4000 - 10^8$	10000	9999.6
	$\varepsilon = 0 - 0.05$	100	$500 \cdot 10^{-6}$
III	$Re = 4000 - 10^8$	100	999960
	$\varepsilon = 0 - 0.05$	10000	$5 \cdot 10^{-6}$

It should be noted that similar analysis covering the observed range ($Re = 4000 - 10^8$ and $\varepsilon = 0 - 0.05$) with a much lesser number of points (about 500 points in [20], 1000 points in [21], 10000 points in [5] and [22], 740 points in the recent one [7]).

The statistical comparison of different equations was also carried out using the "Model selection criterion" (*MSC*) and "Akaike Information Criterion" (*AIC*).

Table 3. Statistical parameters for observed equations

Eq. num.	*meanRe* [%]	$maxRe^{+}$ [%]	$maxRe^{-}$ [%]	Θ [%]	Δ_{av} [%]	*MSC*	$AIC \cdot 10^{-6}$	*NP*	*NC*
(4)	7.517	15.90	– 12.532	84.22	8.853	– 29.92	3.493	4	5
(5)	16.42	46.83	– 2.622	30.26	18.34	– 30.72	4.864	3	4
(6)	3.647	100	– 6.241	99.02	10.37	–	1.040	7	11
(7)	0.0818	0	– 0.00121	100	0.685	–	– 1.882	5	8
(8)	0.181	0.790	– 3.185	100	0.335	– 25.95	– 3.212	5	8
(9)	0.0406	0.708	– 3.358	100	0.315	–	– 3.305	5	8
(10)	0.0676	0.316	– 0.324	100	0.0686	– 25.16	– 6.514	8	14
(11)	90.21	94.45	0	0	90.33	– 32.33	7.857	4	7
(12)	0.000612	0.114	– 0.0496	100	0.00615	–	– 14.087	7	16
(13)	0.207	1.420	– 1.314	100	0.222	–	– 4.393	5	8
(14)	16.16	27.30	– 2.622	30.26	17.99	– 30.71	4.864	4	5
(15)	0.0324	0.00404	– 2.729	100	0.245	–	– 3.755	6	10
(16)	0.0680	0.0815	– 0.146	100	0.069	– 25.00	– 6.511	11	20
(17)	0.0550	0.441	– 0.491	100	0.077	– 22.96	– 6.769	8	11
(18)	0.118	3.374	– 1.655	100	0.220	– 25.37	– 4.590	9	16
(19)	0.123	0.124	– 2.856	100	0.280	– 25.33	– 3.530	9	16

The *MSC* and *AIC* attempt to represent the "information content" of a given set of parameter estimates by relating the coefficient of determination to the *NP* (or equivalently, the number of degrees of freedom) that were required to obtain the fit. When comparing two models (equation) with different numbers of parameters, this criterion places a burden on the model with more parameters not only to have a better coefficient of determination, but quantifies how much better it must be for the model to be deemed more appropriate.

MSC criterion is given in the form

$$MSC = \ln\left[\frac{\sum_{i=1}^{n}\left(f_{C,i} - f_{C,av}\right)}{\sum_{i=1}^{n}\left(f_{C,i} - f_{pred,i}\right)}\right] - \frac{2NP}{n} \qquad (26)$$

where *NP* is the number of parameters in proposed equation.

For this criterion, the most appropriate model will be that with the largest *MSC*, because we want to maximize information content of the model.

AIC is defined by the following expression

$$AIC = n\ln\left[\sum_{i=1}^{n}\left(f_{C,i} - f_{pred,i}\right)^2\right] + 2NP\,. \qquad (27)$$

The *AIC* as defined above is dependent on the magnitude of the data points as well as the number of observations. According to this criterion, the most appropriate model is the one with the smallest values of the *AIC*. Statistical comparison of equations (4) – (19) with Colebrook's equation (3) is given in Table 3, where *NC* is the number of mathematical calculations in a given equation.

The numbers from Table 3 speak for themselves. Equation (12) is the best one according to most important criterions Δ_{av} and Θ, and maximal relative errors are quite low. The only shortcoming of the (12) is the number of calculations (mathematical operations) that have to be done in order to obtain the result. It is interesting to compare, for example, (10) and (16). They have almost the same standard Δ_{av} and Θ, as well as other statistical parameters. Equation (10) should be given the advantage, in hand calculations, because it has much lesser *NP* and *NC* compared to (16).

Another interesting equation is (13). Although it is published 28 years ago, it provides very fine statistical parameters and needs only *NC* = 8 mathematical operations.

Altshul's equation (5) and Tsal's correction (14) of Altshul's equation is cited in one of the most significant engineering handbooks [22]. The citation from [22] is interesting: "Friction factors obtained from the Altshul-Tsal equation are within 1.6 % of those obtained by Colebrook's equation."

Our analysis shows that both equations do not predict friction factor well. Maximal relative error of (14) is 27.30 %, standard deviation is about 18 %. Alshul's, equation shows even worse parameters: maximal error 46.83 % is highly unacceptable. Although *NC* is small, these equations cannot be recommended for engineering practice. Equation (11) is the worst one among the cited equations.

4. CONCLUSION

As stated by many engineers and scientists, famous Colebrook's equation is still the best equation that provides a link between the friction factor, Reynolds number and relative roughness. Its only disadvantage is the implicit form of equation, and many authors reported their explicit approximations.

After the statistical analysis given in this paper, two equations can be recommended:

- equation (12) of Zigrang and Sylvester [16] provides the most accurate value of friction factor using 16 calculations to obtain the result;
- equation (13) of Haaland [17] provides reasonably good statistical parameters but needs only 8 calculations, which is more convenient for hand calculation.

Equations (4) – (6), (11) and (14) should be avoided in engineering practice.

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NOMENCLATURE

ε	relative pipe roughness
f	friction factor
n	number of nodes (points)
NC	number of mathematical calculations
NP	number of parameters
Re	Reynolds number

Greek symbols

β	nondimensional parameter
Δ	standard deviation
Θ	correlation ratio

Subscripts

av	average
C	Colebrook
pred	predicted

ПРЕГЛЕД ЕКСПЛИЦИТНИХ АПРОКСИМАЦИЈА КОЛБРУКОВЕ ЈЕДНАЧИНЕ ЗА КОЕФИЦИЈЕНТ ТРЕЊА

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У раду је дат преглед најчешће коришћених експлицитних једначина за одређивање коефицијента трења у глатким и храпавим цевима. Одступање наведених једначина од Колбрукове једначине изражено је преко средње релативне грешке, максималне позитивне грешке, максималне негативне грешке, средњег одступања и корелационог односа. Осим наведених критеријума, поређење једначина је извршено и коришћењем „Model selection criterion" (MSC) и „Akaike Information Criterion" (AIC). Наведеном анализом установљено је да су одступања једначине коју су предложили Зигранг и Силвестер најмања у односу на Колбрукову релацију, а да је Халандова једначина најпогоднија за инжењерску употребу.