Nina M. Anđelić
Associate Professor
University of Belgrade
Faculty of Mechanical Engineering

Torsional Analysis of Open Section Thin-Walled Beams

The main purpose of this paper is to present one approach to the optimization of thin-walled I, Z and channel-section beams subjected to constrained torsion. The displacement constraints are introduced: allowable angle of twist and allowable angle of twist per unit length. The area of the cross-section is assumed to be the objective function. Applying the Lagrange multiplier method, the equations whose solutions represent the optimal values of the ratios of the parts of the chosen cross-sections are derived.

***Keywords**: thin-walled beams, cantilever beam, optimal dimensions, displacement constraints.*

1. INTRODUCTION

Many studies have been made on the optimization problems treating the cases where geometric configurations of structures are specified and only the dimensions of members, such as areas of members' cross-sections, are determined in order to attain the minimum structural weight or cost [1-5]. Many methods have been developed for the determination of the local minimum point for the optimization problem [6-8]. Very often used types of cross-sections, particularly in steel structures are the I, Z and channel sections.

A series of works appear where the optimization parameters of various cross-sections, such as I-section [9], channel-section [10] or Z-section beams [11] have been determined by Lagrange's multipliers method.

The starting points during the formulation of the basic mathematical model are the assumptions of the thin-walled beam theory, on one hand [12, 13], and the basic assumptions of the optimum design on the other [1-5].

Open thin-walled steel sections subjected to twisting moments are generally prone to large warping stresses and excessive angles of twist. It is therefore a common practice to avoid twisting moments in steel assemblies consisting of steel open sections whenever it is possible. However, in a number of practical applications, twisting cannot be avoided and the designer is compelled to count on the torsional resistance of these members. The classical formulation for open thin-walled sections subjected to torsion was developed by Vlasov [13]. The Vlasov formulation is based on two fundamental kinematic assumptions: (a) In-plane deformations of the section are negligible, and (b) shear strains along the section mid-surface are negligible.

2. BASIC ASSUMPTIONS

The formulation is restricted to the torsional analysis of open section thin-walled beams.

Received: January 2012, Accepted: January 2012
Correspondence to: Dr Nina Anđelić
Faculty of Mechanical Engineering,
Kraljice Marije 16, 11120 Belgrade 35, Serbia
E-mail: nandjelic@mas.bg.ac.rs

The considered cantilever beam, of the length *l* is subjected to the constrained torsion because of the fact that its one end is fixed and the other free end is loaded by a concentrated torque M^*. The cross-section (Fig. 1) is supposed to have flanges of mutually equal widths and thicknesses $b_1 = b_3$, $t_1 = t_3$.

2.1 Objective function

It is very important to find out the optimal dimensions of cross-sections. The process of selecting the best solution from various possible solutions must be based on a prescribed criterion known as the objective function. In the considered problem the cross-sectional area will be treated as an objective function. The aim of the paper is to determine the minimal mass of the beam or, in another way, to find the minimal cross-sectional area

$$A = A_{\min} \tag{1}$$

for the given loads and material and geometrical properties of the considered beam.

It is obvious from the Fig. 1 that

$$A = \sum b_i t_i , \quad i = 1,2,3 \tag{2}$$

where b_i and t_i are widths and thicknesses of the parts of the considered cross-sections.

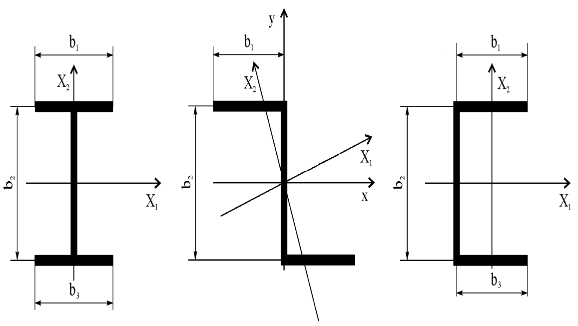

Figure 1. a) I-section; b) Z-section; c) Channel-section

2.2 Constraints

Only the displacements will be taken into account in the calculations that follow and the constraints treated in the paper are the displacement constraints.

The considered displacement constraints are allowable angle of twist and allowable angle of twist per unit length denoted by θ_0 and θ_0' respectively.

The ratio

$$z = b_2 / b_1 \tag{3}$$

will be the optimal relation of the dimensions of the considered cross-sections.

The flexural-torsion cross-section characteristic [5] is given by the expression

$$k = \sqrt{GI_t / EI_\omega} \tag{4}$$

where:

- $\psi = t_2 / t_1$ - the ratios of thickness and length of the parts of the cross-section
- I_t - torsion constant derived for each of the three considered cross-sections [12],
- I_ω - sectorial moment of inertia derived for each of the three considered cross-sections [12],
- E - modulus of elasticity and
- G - shear modulus.

If the constraint is allowable angle of twist θ_0, the constraind function (5)

$$\theta_{\max} = \theta(l) = \frac{M^* l}{GI_t}\left(1 - \frac{Thkl}{kl}\right) \le \theta_0 \tag{5}$$

can be written as

$$\varphi_1 = kl - Thkl - \frac{GI_t kl}{M^* l}\theta_0 \le 0 . \tag{6}$$

If the constraint is allowable angle of twist per unit length θ_0', the constraind function (7)

$$\theta'_{\max} = \theta'(l) = \frac{M^*}{GI_t}\left(1 - \frac{1}{Chkl}\right) \le \theta'_0 \tag{7}$$

can be written as

$$\varphi_2 = Chkl\left(1 - \theta'_0 \frac{GI_t}{M^*}\right) - 1 \le 0 . \tag{8}$$

2.3 Lagrange multiplier method

The Lagrange multiplier method [2,3,14,15] is a powerful tool for solving this class of problems and represents the classical approach to the constraint optimization.

Applying the Lagrange multiplier method to the vector which depends on two parameters b_i, i = 1,2, the system of equations $\varphi(b_i) = 0$, i = 1,2 will be obtained

$$\frac{\partial}{\partial b_i}\left[A(b_1, b_2) + \lambda\, \phi(b_1, b_2)\right] = 0, \quad (\text{i} = 1,2) \tag{9}$$

and after the elimination of the multiplier λ from (9), it becomes (10)

$$\frac{\partial A(b_1,b_2)}{\partial b_1} \cdot \frac{\partial \phi(b_1,b_2)}{\partial b_2} = \frac{\partial A(b_1,b_2)}{\partial b_2} \cdot \frac{\partial \phi(b_1,b_2)}{\partial b_1} \tag{10}$$

3. ANALYTICAL APPROACH

The diagrams of sectorial coordinates [5] are shown in the Fig.2 for each cross-section separately, where P is the shear center.

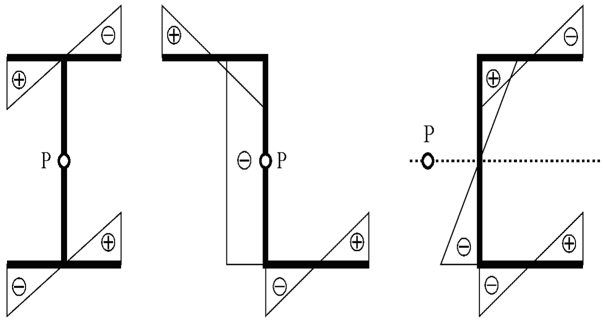

Figure 2. Sectorial coordinates for the I, Z and Channel cross-section

The expressions of torsion constant and sectorial moment of inertia are derived for each of three considered cross-sections according to (11)

$$I_t = \frac{1}{3}\sum b_i t_i^3 , \quad I_\omega = \int_A \omega^2 dA \tag{11}$$

3.1 I cross-section

The expressions of torsion constant and sectorial moment of inertia for the I-section are:

$$I_t = \frac{1}{3} b_1 t_1^3 \left(2 + \psi^3 z\right) , \; I_\omega = \frac{1}{24} b_1^3 b_2^2 t_1 \tag{12}$$

Applying the Lagrange multiplier method, the equations (13) according to allowable angle of twist (θ_0), and (14) according to allowable angle of twist per unit length (θ_0') are obtained:

- θ_0 – constraint:

$$\begin{aligned} 2\frac{Gl}{Ek}\frac{t_1^2}{b_1^2 b_2^3}&\left[-3Th^2 kl + \frac{G\theta_0}{M^* l} b_1 t_1^3 \left(2 + \psi^3 z\right)\right] \cdot \\ &\cdot \left(8 - 4\psi z + 2\psi^3 z - 3\psi^4 z^2\right) - \\ &- \frac{G\theta_0}{M^* l} kl t_1^3 \left(\psi^2 - 1\right)\psi = 0 \end{aligned} \tag{13}$$

- θ_0' – constraint:

$$\begin{aligned} 2Thkl \frac{1}{b_1^3 b_2^2 t_1}\frac{Gl}{Ek}&\left[3 - \frac{G\theta'_0}{M^*} b_1 t_1^3 \left(2 + \psi^3 z\right)\right] \cdot \\ &\cdot \left(-8 + 4\psi z - 2\psi^3 z + 3\psi^4 z^2\right) + \\ &+ \frac{G\theta'_0}{M^*}\left(1 - \psi^2\right)\psi z = 0. \end{aligned} \tag{14}$$

In the considered case when the I – beam is the object of the optimization the equations (13) and (14), combined with (6) and (8) are reduced to the equation

$$\sum_{i=0}^{2} c_i z^{\,i} = 0 \tag{15}$$

where the coefficients c_i are given in the form (16) if the constraint is allowable angle of twist (θ_0), i.e. (17) if the constraint is allowable angle of twist per unit length(θ_0'):

-θ_0 – constraint:

$$c_0 = 8,$$

$$c_1 = -2\psi\left[2-\psi^2+2\frac{\psi^2-1}{\dfrac{kl-klTh^2kl-Thkl}{kl-Thkl}}\right],$$

$$c_2 = -3\psi^4. \tag{16}$$

-θ_0' – constraint:

$$c_0 = -8,$$

$$c_1 = 2\psi\left[2-\psi^2+2\frac{\psi^2-1}{\dfrac{klThkl}{1-Chkl}}\right],$$

$$c_2 = 3\psi^4. \tag{17}$$

3.2 Z cross-section

The expressions of torsion constant and sectorial moment of inertia for the Z-section are:

$$I_t = \frac{1}{3}b_1t_1^3\left(2+\psi^3 z\right), I_\omega = \frac{1}{12}b_1^3b_2^2\frac{1+2\psi z}{2+\psi z} \tag{18}$$

Applying the Lagrange multiplier method, the equations (19) and (20) are obtained

-θ_0 – constraint:

$$4\frac{Gl}{Ek}\frac{t_1^2}{b_1^2b_2^2}\cdot\left(\frac{8+22\psi z+2\psi^3 z+\psi^2z^2+5\psi^4z^2}{(1+2\psi z)^2}+\right.$$
$$\left.+\frac{-4\psi^3z^3-4\psi^5z^3-3\psi^6z^4}{(1+2\psi z)^2}\right)\cdot$$
$$\cdot\left[-Th^2kl+\frac{1}{3}\frac{G\theta_0}{M^*l}b_1t_1^3\left(2+\psi^3z\right)\right]-$$
$$-\frac{2}{3}\frac{G\theta_0}{M^*l}klb_1t_1^3\left(\psi^2-1\right)\psi z = 0. \tag{19}$$

-θ_0'– constraint:

$$4Thkl\frac{Gl}{Ek}\frac{t_1^2}{b_1^2b_2^2}\cdot$$
$$\cdot\left(\frac{8+22\psi z+2\psi^3 z+\psi^2z^2+5\psi^4z^2}{(1+2\psi z)^2}+\right.$$
$$\left.+\frac{-4\psi^3z^3-4\psi^5z^3-3\psi^6z^4}{(1+2\psi z)^2}\right)\cdot$$
$$\cdot\left[-1+\frac{1}{3}\frac{G\theta_0'}{M^*}b_1t_1^3\left(2+\psi^3z\right)\right]-$$
$$-\frac{2}{3}\frac{G\theta_0'}{M^*}b_1t_1^3\left(\psi^2-1\right)\psi = 0 \tag{20}$$

In the considered case when the Z-beam is the object of the optimization the equations (19) and (20), combined with (6) and (8) are reduced to the equation

$$\sum_{i=0}^{4} c_i z^{\,i} = 0\,, \tag{21}$$

where the coefficients c_i are:

-θ_0 – constraint:

$$c_0 = 8,$$

$$c_1 = 2\psi\left[11+\psi^2-2\frac{\psi^2-1}{1-\dfrac{klTh^2kl}{kl-Thkl}}\right],$$

$$c_2 = \psi^2\left[1+5\psi^2-10\frac{\psi^2-1}{1-\dfrac{klTh^2kl}{kl-Thkl}}\right],$$

$$c_3 = -4\psi^3\left[1+\psi^2+\frac{\psi^2-1}{1-\dfrac{klTh^2kl}{kl-Thkl}}\right],$$

$$c_4 = -3\psi^6. \tag{22}$$

-θ_0' – constraint:

$$c_0 = -8,$$

$$c_1 = -2\psi\left[11+\psi^2-2\frac{\psi^2-1}{\dfrac{klThkl}{1-Chkl}}\right],$$

$$c_2 = -\psi^2\left[1+5\psi^2-10\frac{\psi^2-1}{\dfrac{klThkl}{1-Chkl}}\right],$$

$$c_3 = 4\psi^3\left[1+\psi^2+\frac{\psi^2-1}{\dfrac{klThkl}{1-Chkl}}\right],$$

$$c_4 = 3\psi^6. \tag{23}$$

3.3 Channel cross-section

The expressions of torsion constant and sectorial moment of inertia for the channel-section are :

$$I_t = \frac{1}{3} b_1 t_1^3 \left(2 + \psi^3 z\right),$$
$$I_\omega = \frac{1}{12} b_1^3 b_2^2 \frac{1 + 2\psi z}{2 + \psi z}. \tag{24}$$

Applying the Lagrange multiplier method, the equations (25) and (26) are obtained

- θ_0 – constraint:

$$4 \frac{Gl}{Ek} \frac{t_1^2}{b_1^2 b_2^2} \cdot$$
$$\cdot \left(\frac{72 + 42\psi z + 18\psi^3 z - 13\psi^2 z^2 - 3\psi^4 z^2}{(3 + 2\psi z)^2} + \right.$$
$$\left. + \frac{-4\psi^3 z^3 - 16\psi^5 z^3 - 3\psi^6 z^4}{(3 + 2\psi z)^2} \right) \cdot$$
$$\cdot \left[-Th^2 kl + \frac{1}{3} \frac{G\theta_0}{M^* l} b_1 t_1^3 \left(2 + \psi^3 z\right) \right] -$$
$$- \frac{2}{3} \frac{G\theta_0}{M^* l} kl \cdot \psi z \cdot b_1 t_1^3 \left(\psi^2 - 1\right) = 0. \tag{25}$$

- θ_0' – constraint:

$$2Thkl \frac{Gl}{Ek} \frac{t_1^2}{b_1^2 b_2^2} \cdot$$
$$\cdot \left(\frac{-72 - 42\psi - 18\psi^3 z + 13\psi^2 z^2 + 3\psi^4 z^2}{(3 + 2\psi z)^2} + \right.$$
$$\left. + \frac{+4\psi^3 z^3 + 16\psi^5 z^3 + 3\psi^6 z^4}{(3 + 2\psi z)^2} \right) \cdot$$
$$\cdot \left[1 - \frac{1}{3} \frac{G\theta_0'}{M^*} b_1 t_1^3 \left(2 + \psi^3 z\right) \right] +$$
$$+ \frac{2}{3} \frac{G\theta_0'}{M^*} \psi z b_1 t_1^3 \left(1 - \psi^2\right) = 0. \tag{26}$$

In the considered case when the channel-section is the object of the optimization the equations (25) and (26), combined with (6) and (8) are reduced to the equation

$$\sum_{i=0}^{4} c_i z^i = 0 \tag{27}$$

- θ_0 – constraint:

$$c_0 = 72,$$
$$c_1 = 6\psi \left[7 + 3\psi^2 - 6 \frac{\psi^2 - 1}{1 - \frac{klTh^2 kl}{kl - Thkl}} \right],$$
$$c_2 = -\psi^2 \left[13 + 3\psi^2 + 30 \frac{\psi^2 - 1}{1 - \frac{klTh^2 kl}{kl - Thkl}} \right],$$
$$c_3 = -4\psi^3 \left[1 + 4\psi^2 + \frac{\psi^2 - 1}{1 - \frac{klTh^2 kl}{kl - Thkl}} \right],$$
$$c_4 = -3\psi^6. \tag{28}$$

- θ_0' – constraint:

$$c_0 = -72,$$
$$c_1 = -6\psi \left[7 + 3\psi^2 - 6 \frac{\psi^2 - 1}{\frac{klThkl}{1 - Chkl}} \right],$$
$$c_2 = \psi^2 \left[13 + 3\psi^2 + 30 \frac{\psi^2 - 1}{\frac{klThkl}{1 - Chkl}} \right],$$
$$c_3 = 4\psi^3 \left[1 + 4\psi^2 + \frac{\psi^2 - 1}{\frac{klThkl}{1 - Chkl}} \right],$$
$$c_4 = 3\psi^6. \tag{29}$$

4. ANALYSIS AND DISCUSSIONS

The following expressions will be introduced

$$D = \frac{\psi^2 - 1}{\frac{kl - Thkl - klTh^2 kl}{kl - Thkl}},$$
$$D_1 = \frac{\psi^2 - 1}{\frac{klThkl}{1 - Chkl}}. \tag{30}$$

The calculation is made for the cantilever beam of chosen section of the length $0.25 \le l \le 200$ cm. Values kl are calculated using data for standard profiles and ratio $\psi = t_2/t_1$ is taken as $\psi = 0.5$; 0.75; 1.

The results for ratios (3) $z = b_2/b_1$ are presented graphically in Figs. 3-5.

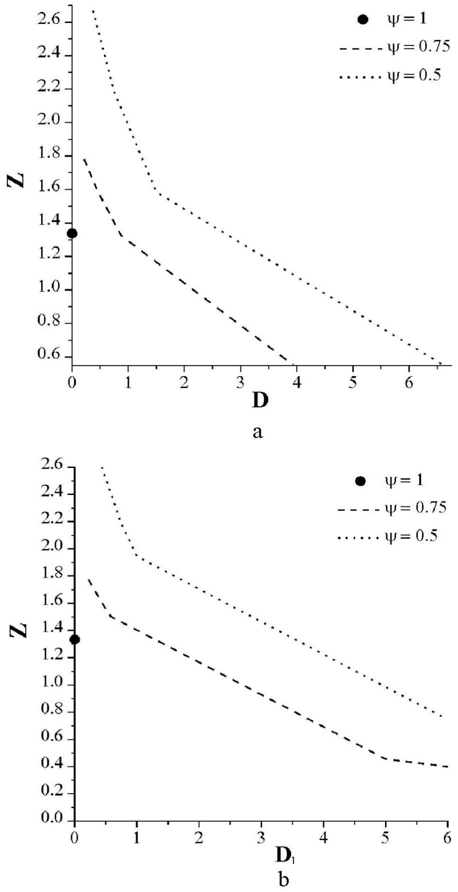

Figure 3. I-beam - the optimal ratios: a) θ_0 – constraint, b) θ_0'-constraint

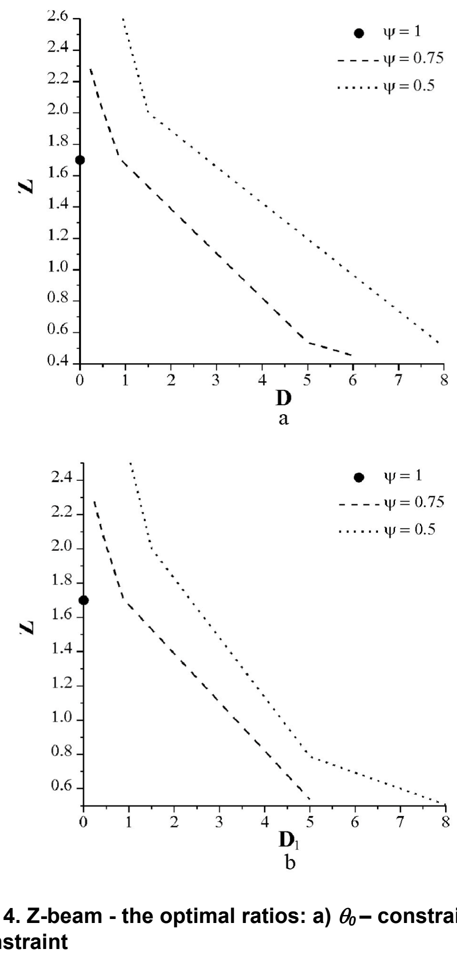

Figure 4. Z-beam - the optimal ratios: a) θ_0 – constraint, b) θ_0'-constraint

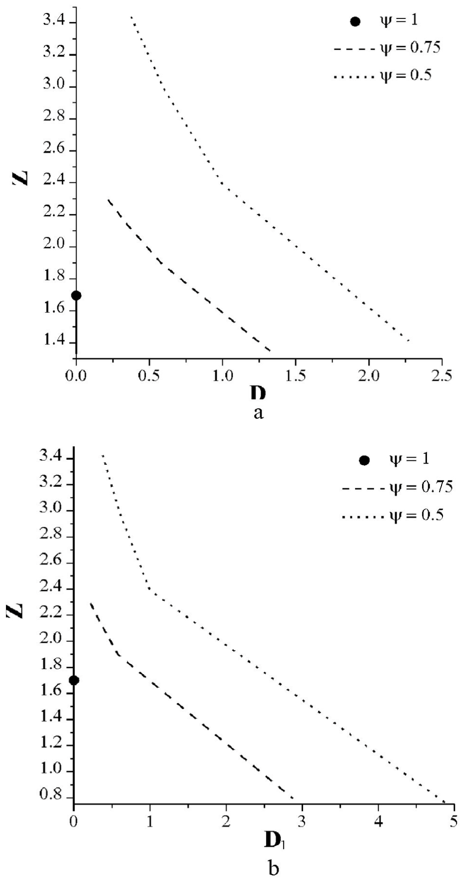

Figure 5. Channel-section beam - the optimal ratios: a) θ_0 – constraint, b) θ_0'-constraint

After the calculations, it can be concluded that the increase of the expressions D, e.i. D_1 will decrease the optimal relations z.

I – beam:

- Optimal values z for strain constraint θ_0

- $\psi=1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.33$,
- $\psi=0.75 \Rightarrow 0.22 \le D_1 \le 437.5 \Rightarrow 1.78 \ge z \ge 0$,
- $\psi=0.5 \Rightarrow 0.38 \le D_1 \le 750 \Rightarrow 2.67 \ge z \ge 0$.

The calculations show that the optimal values of z for the I-section beam are very small for the lengths $l > 100$ cm. Because of that it is possible to say that the application of this criterion makes sense for following lengths:

- for $\psi=0.5$: $l \approx 90\ cm \Rightarrow z \ge 0.51$ and
- for $\psi=0.75$: $l \approx 95\ cm \Rightarrow z \ge 0.45$.

- Optimal values **z** for strain constraint θ_0'

The optimal values of z are for the lengths $l \le 100$ cm

- $\psi=1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.33$,
- $\psi=0.75$: $l \approx 95\ cm, D_1 \le 5 \Rightarrow z \ge 0.4549$,
- $\psi=0.5$: $l \approx 90\ cm, D_1 \le 7 \Rightarrow z \ge 0.5049$.

For $l > 100$ cm: $z = b_2 / b_1 < 0.4$.

Z – beam:

- Optimal values z for strain constraint θ_0

- $\psi=1 \Rightarrow D_1 = 0 \Rightarrow z = \text{const} = 1.72$,
- $\psi=0.75$: $l \approx 150\ cm, D \le 6 \Rightarrow z \ge 0.4532$,
- $\psi=0.5$: $l \approx 120\ cm, D \le 8 \Rightarrow z \ge 0.50688$.

The optimal values of z are for the lengths $l \leq 150$ cm.

- Optimal values z for strain constraint $\theta_0^{'}$

- $\psi=1 \Rightarrow D_1 = 0 \Rightarrow z$ = const = 1.72,

- $\psi=0.75$: $l \approx 48$ cm, $D_1 \leq 5 \Rightarrow z \geq 0.5372$,

- $\psi=0.5$: $l \approx 47$ cm, $D_1 \leq 8 \Rightarrow z \geq 0.50688$.

The optimal values of z are for the lengths $l \leq 50$ cm.

Channel section – beam:

Calculations show that optimal values z, for the constraints θ_0 and $\theta_0^{'}$ are for the lengts $l \leq 200cm$.

5. CONCLUSION

This paper presents an approach to the optimization of thin-walled open-section cantilever beams, using the Lagrange multiplier method. Selecting the cross-section area as the objective function and deformation constrains for constraint functions, optimal ratios of cross-section individual parts (webs and flanges) are determined.

Based on the obtained results (Figs. 3, 4 and 5), it can be seen that some differences exist between coefficients c_i calculated using the criteria θ_0 or $\theta_0^{'}$, and a minimum disagreement between obtained values for z is observed. Optimal values z obtained by using criterion the θ_0 are slightly higher than values obtained by the $\theta_0^{'}$ criterion.

On the bases of the proposed optimization procedure, it is possible to calculate the optimal ratios between the parts of the considered thin-walled profiles in the a very simple way.

ACKNOWLEDGMENT

This work is a contribution to the Ministry of Science and Technological Development of Serbia funded project TR 35040 and TR 35011.

REFERENCES

[1] Onwubiko, C.: *Introduction to Engineering Design Optimization*, Prentice Hall, New Jersey Inc, 2000.

[2] Fox, R.L.: *Optimization Methods for Engineering Design*, Addison-Wesley Publishing Company Inc, 1971.

[3] Rozvany, G.: *Shape and Layout Optimization of Structural Systems and Optimality Criteria Methods*, CISM, No.325, Springer-Verlag, Wien, 1992.

[4] Brousse, P.: *Structural Optimization*, CISM, No.237, Springer-Verlag, Wien, 1975.

[5] Prager, W.: *Introduction to Structural Optimization*, CISM, No.212, Springer-Verlag, Wien, 1974.

[6] Farkas, J.: *Optimum Design of Metal Structures*, Akademiai KIADO, Budapest, 1984.

[7] Tian Y.S. and Lu, T.J. : Minimum weight of cold-formed steel sections under compression, Thin-walled structures, Vol. 42, No. 4, pp. 515-532, 2004.

[8] Mijailović, R.: Optimum design of lattice-columns for buckling, Structural and Multidisciplinary Optimization, Vol. 42, No. 6, pp. 897-906, 2010.

[9] Andjelić, N.: Thin walled I-beam under complex loads - Optimization according to stress constraint, FME Transactions, Vol. 3, No. 2, pp. 55 – 60, 2003.

[10] Andjelić, N. and Milošević-Mitić V.: An approach to the optimization of thin-walled cantilever open section beams, Theoretical and applied mechanics, Vol. 34, No. 4, pp. 323-340, 2007.

[11] Andjelić, N., Milošević-Mitić V. and Maneski, T.: An approach to the optimization of Thin-walled Z-beam, Strojniški vestnik - Journal of Mechanical Engineering, Vol. 55, No. 12, pp. 742-748, 2009.

[12] Kollbruner C.F. and Hajdin, N.: *Dunnwandige Stabe*, Band 1, Springer Verlag, 1972.

[13] Vlasov, V.Z.: *Thin-Walled Elastic Beams*, second ed., Moscow, 1959, (English translation, Israel Program for Scientific Translation, Jerusalem), 1961.

[14] Bertsekas, D.P.: *Constrained Optimization and Lagrange Multiplier Methods*, Athena Scientific, Belmont, 1996.

[15] Zoller, K.: Zur anschaulichen Deutung der Lagrangeschen Gleichungen zweiter Art, Ingenieur-Archiv, Vol. 41, No. 4, pp. 270-277, 1972.

ТАНКОЗИДИ ОТВОРЕНИ ПОПРЕЧНИ ПРЕСЕЦИ ИЗЛОЖЕНИ ОГРАНИЧЕНОЈ ТОРЗИЈИ

Н. Анђелић

Основни циљ овог рада је да прикаже један приступ оптимизацији танкозидних I, Z и U конзолних конструкционих елемената отворених попречних пресека изложених ограниченој торзији. За критеријум ограничења одабран је критеријум ограничења деформација: дозвољени угао увијања и дозвољене угао увијања по јединици дужине. За функцију циља одабрана је површина попречног пресека носача. Применом методе Лагранжовог множитеља изведене су једначине чија решења представљају оптималне односе димензија попречног пресека изабраног облика.