On Brachistochronic Motion of a Multibody System with Two Degrees of Freedom with Real Constraints

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The paper considers a case of brachistrochronic motion of the mechanical system in the field of conservative forces, subject to the action of constraints with Coulomb friction. In this case an analogy is made between the two approaches of solving this problem of the mechanical system with two degrees of freedom. The mathematical model used to compute the brachistohrone in this special case of the multibody system with two deegres of freedom is based on varational calculus. The complete analogy is made with a solution in relation to material point.

Keywords : *Brachistochronic motion, Coulomb friction.*

1. INTRODUCTION

The Bernoulli's case of brachistochronic motion of a particle (cf. [1]) was for the first time extended by Euler in [2] . Euler considered motion of a particle in resisting medium in which the force of resistance depends only on velocity of the particle.

The case of brachistochronic motion of a particle in the field of gravity subject to the action of Coulomb friction solved by Ashby in [3] presents a special case in which the power of friction forces is described as linear function in relation to generalized velocities and accelerations. The brachistochronic motion of a particle along a rough surface was treated by Čović and Vesković in [4] where the power of friction forces is given as the sum of two functions, one in relation to generalized coordinates and velocities and the other function which is linear in relation to accelerations and whose coefficients depend on generalized coordinates and velocities.

The Bernoulli's brachistochrone problem was extended to the system of rigid bodies by Čović and Vesković in [5].

The brachistochronic motion of a mechanical system with two degrees of freedom subject to the action of constraints with Coulomb friction presented by Djuric in [6] use the form of the power of friction forces given in [4] but in case when the power of friction forces, potential and kinetic energy does not depend on generalized coordinates. In the paper [7] Čović and Vesković showed complete analogy between the brachistochronic motion of a mechanical system with two degrees of freedom and the brachistochronic motion of a particle under friction forces where the power of friction forces is expressed as function in relation to generalized velocities and accelerations, in a more general form than in [3] and [4] .

In the paper [8] Šalinić, Obradović, and Mitrović considered the case of brachistochronic motion of mechanical system with two degrees of freedom in which ideal bilateral constraints and one unilateral constrain with Coulomb friction are imposed on the system.

In this paper, the choice of the functions of generalized velocities in expression for power of Coulomb friction shows that in case of mechanical system with two degrees of freedom of motion the complete parameterization of differential equations of motions is similar to Ashby's frictional brachistochrone in relation to material particle. However introduced functions are different than in the papers mentioned above and this leads to new differential equations of brachistochronic motion which are solved.

2. FORMULATION OF THE PROBLEM

We consider, in this paper, the motion of a mechanical system in a stationary field of potential forces with potential $\overline{\Pi}=\overline{\Pi}(\overline{q})$, subject to the action of real constraints. The configuration of the system is defined by the set of Lagrangian coordinates $\overline{q}=(\overline{q}^1,\overline{q}^2,...,\overline{q}^n)$, to which correspond the generalized velocities $\dot{\overline{q}}=(\dot{\overline{q}}^1,\dot{\overline{q}}^2,...,\dot{\overline{q}}^n)$ Lagrangian function of the system has the form ([2])

$$\overline{L}(\overline{q},\dot{\overline{q}})=\overline{T}(\overline{q},\dot{\overline{q}})-\overline{\Pi}(\overline{q}) \tag{1}$$

where $\overline{T}$ is kinetic energy of the system

$$\overline{T}=\frac{1}{2}a_{\alpha\beta}(\overline{q})\dot{\overline{q}}^{\alpha}\dot{\overline{q}}^{\beta},\ \alpha,\beta=1,2,...,n. \tag{2}$$

Differential equations of motions of the mechanical system have the well known form (cf.[9])

$$\frac{d}{dt}\frac{\partial\overline{L}}{\partial\dot{\overline{q}}^{\alpha}}-\frac{\partial\overline{L}}{\partial\overline{q}^{\alpha}}=\overline{Q}_{\alpha}^{\mu}+R_{\alpha}, \tag{3}$$

where $\overline{Q}_{\alpha}^{\mu}$ are generalized forces of Coulomb friction and R_{α} are generalized control forces. Let us assume that initial position of the system is defined by

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the set of given coordinates $\bar{q}_0^{\alpha}$ at moment $t = t_0$, which is set in advance, where it was at rest and let final position is defined by the set of coordinates $\bar{q}_1^{\alpha}$ at moment $t = t_1$, which is unknown. The time the system needs to move from initial to final position is determined by relation

$$I = \int_0^{t_1} dt. \tag{4}$$

If we assume that the system moves from initial to final configuration along one definite trajectory for which Eqs. (4) has minimum value

$$I = \int_0^{t_1} dt \to \inf \tag{5}$$

we will consider brachistochronic motion. If we now introduce Bernoulli's condition's (cf.[3]), i.e. the conditions which do not disturb the principle of work and energy subject to the action of control forces in virtue of $R_i \dot{q}^i = 0$, we formulate variational problem as constrained with constraint which represents the principle of work and energy

$$\dot{\bar{T}} = \bar{P}^{\mu} - \dot{\Pi} \;\Rightarrow\; \dot{\bar{T}} + \dot{\bar{\Pi}} - \bar{P}^{\mu} = 0, \tag{6}$$

where power of generalized forces of Coulomb friction has the form

$$\bar{P}^{\mu} = \bar{Q}_{\alpha}^{\mu} \dot{\bar{q}}^{\alpha}, \tag{7}$$

so that relation (5) becomes

$$I_1 = \int_0^{t_1} F dt \to \inf ., \tag{8}$$

where

$$F(\lambda, \bar{q}, \dot{\bar{q}}, \ddot{\bar{q}}) = 1 + \lambda \left(\dot{\bar{T}} + \dot{\bar{\Pi}} - \bar{P}^{\mu} \right). \tag{9}$$

3. GENERAL PART

Let us consider a case of brachistochronic motion presented in [7] in which the power of forces of Coulomb friction (cf. (7)) has the form:

$$\bar{P}^{\mu} = \bar{\psi}(\bar{q}, \dot{\bar{q}}) + \bar{\varphi}_{\beta}(\bar{q}, \dot{\bar{q}}) \ddot{\bar{q}}^{\beta}, \tag{10}$$

and let us assume now that functions $\bar{\psi}(\bar{q}, \dot{\bar{q}})$ and $\bar{\varphi}_{\beta}(\bar{q}, \dot{\bar{q}})$ have the form

$$\bar{\psi}(\bar{q}, \dot{\bar{q}}) = -b_{\alpha}(\bar{q}) \dot{\bar{q}}^{\alpha}, \quad \bar{\phi}_{\beta}(\bar{q}, \dot{\bar{q}}) = -d_{\alpha\beta}(\bar{q}) \dot{\bar{q}}^{\alpha}, \tag{11}$$

where the following

$$\bar{\psi}(\bar{q}, \dot{\bar{q}}),\ \bar{\varphi}_{\beta}(\bar{q}, \dot{\bar{q}}) \in C^2, \tag{12}$$

holds.

Our aim is to obtain differential equations of brachistochrinic motion of the system. For that reason, we are able to minimize the integral (4) but in order to avoid second order functional we introduce the following constraints in terms of varational calculus

$$\dot{\bar{q}}^{\alpha} - u^{\alpha} = 0, \quad 2\bar{T} - a_{\alpha\beta} u^{\alpha} u^{\beta} = 0, \tag{13}$$

and the integrand of functional (9) gets the form

$$F = 1 + \lambda \left(\dot{\bar{T}} + \frac{\partial \bar{\Pi}}{\partial \bar{q}^{\alpha}} u^{\alpha} - \tilde{\psi} - \tilde{\phi}^{\beta} \dot{u}^{\beta} \right) +$$

$$+ \sigma_{\alpha} (\dot{\bar{q}}^{\alpha} - u^{\alpha}) + \theta \left(2\bar{T} - a_{\alpha\beta} u^{\alpha} u^{\beta} \right). \tag{14}$$

where

$$\tilde{\psi}_{(\dot{q}=u^{\alpha})} = -b_{\alpha} u^{\beta}, \ \tilde{\varphi}^{\beta}(\dot{q}=u^{\alpha}) = -d_{\alpha\beta} u^{\alpha}, \tag{15}$$

and where $\lambda = \lambda(t)$, $\theta = \theta(t)$ and $\sigma_{\alpha} = \sigma_{\alpha}(t)$ are Lagrange's multipliers.

Assuming that conditions

$$\frac{\partial F}{\partial \bar{q}^{\alpha}} = 0, \tag{16}$$

are further satisfied (cf.[6]), we shall apply transformation to coordinates

$$\bar{q}^{\alpha} = k_{\gamma}^{\alpha} q^{\gamma}, \quad k_{\gamma}^{\alpha} = const., \quad \delta_{\gamma\pi} = a_{\alpha\beta} k_{\gamma}^{\alpha} k_{\pi}^{\beta}, \tag{17}$$

where $\delta_{\gamma\pi}$ is Kroneker delta simbol.

This transformation leads to a new integrand of functional (8)

$$\begin{aligned} F^* = 1 + \lambda \left(\dot{T}^* + c_{\gamma}^* \omega^{\gamma} + b_{\gamma}^* \omega^{\gamma} + d_{\pi\gamma}^* \omega^{\pi} \dot{\omega}^{\gamma} \right) + \\ + \theta \left(2T^* - \delta_{\gamma\pi} \omega^{\gamma} \omega^{\pi} \right) + \sigma_{\gamma}^* \left(\dot{q}^{\gamma} - \omega^{\gamma} \right), \end{aligned} \tag{18}$$

where

$$\begin{aligned} & T^* = \frac{1}{2} \delta_{\gamma\pi} \omega^{\gamma} \omega^{\pi}, \quad c_{\gamma}^* = c_{\alpha} k_{\gamma}^{\alpha}, \\ & b_{\gamma}^* = b_{\alpha} k_{\gamma}^{\alpha}, \quad d_{\pi\gamma}^* = d_{\alpha\beta} k_{\pi}^{\alpha} k_{\gamma}^{\beta}, \\ & \dot{q}^{\gamma} = \omega^{\gamma}, \qquad \sigma_{\gamma}^* = \sigma_{\alpha} k_{\gamma}^{\alpha}. \end{aligned} \tag{19}$$

Euler's equations for (18) are given in [6].

4. MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM

Let us consider a special case of motion of mechanical system with two degrees of freedom. Assuming that condition (16) is further satisfied and having in mind (17)

$$\bar{q}^1 = k_1^1 q^1 + k_2^1 q^2, \quad \bar{q}^2 = k_1^2 q^1 + k_2^2 q^2, \tag{20}$$

where

$$\begin{aligned} & k_1^1 = \frac{1}{\sqrt{a}}, \ k_2^1 = \frac{1}{\sqrt{b}}, \quad k_1^2 = k_1^1, \ k_2^2 = s k_2^1, \\ & a = a_{11} + 2a_{12} + a_{22}, \quad s = -\frac{a_{11} + a_{12}}{a_{12} + a_{22}}, \\ & b = a_{11} + 2s a_{12} + s^2 a_{22}, \ a_{12} + a_{22} \neq 0. \end{aligned} \tag{21}$$

Taking into account (21), kinetic energy of the system considered (cf. (2)) can be written in the form

$$T = \frac{1}{2}V^2, \quad V^2 = \dot{q}_1^2 + \dot{q}_2^2. \tag{22}$$

Potential energy (cf. (16)) get the form

$$\Pi^* = c_1^* q^1 + c_2^* q^2, \tag{23}$$

where (cf. (19), (21))

$$c_1^* = \frac{1}{\sqrt{a}}(c_1 + c_2), \quad c_2^* = \frac{1}{\sqrt{b}}(c_1 + s\, c_2). \tag{24}$$

Power of generalized forces of Coulomb friction (cf. (10)) obtain the form (cf. (20))

$$P^{\mu *} = -b_i^* \dot{q}^i - d_{ij}^* \dot{q}^i \ddot{q}^j, \quad i, j = 1, 2, \tag{25}$$

where (cf. (19))

$$\begin{aligned}
&b_1^* = \frac{1}{\sqrt{a}}(b_1 + b_2), \quad b_2^* = \frac{1}{\sqrt{b}}(b_1 - \frac{s_1}{s_2} b_2),\\
&d_{11}^* = \frac{1}{a}\left(d_{11} + d_{12} + d_{21} + d_{22}\right),\\
&d_{12}^* = \frac{1}{\sqrt{a\,b}}[d_{11} + d_{21} - \frac{s_1}{s_2}(d_{12} + d_{22})],\\
&d_{21}^* = \frac{1}{\sqrt{a\,b}}[d_{11} + d_{12} - \frac{s_1}{s_2}(d_{21} + d_{22})],\\
&d_{22}^* = \frac{1}{b}[d_{11} - \frac{s_1}{s_2}(d_{12} + d_{21}) + (\frac{s_1}{s_2})^2 d_{22}],\\
&s_1 = a_{11} + a_{12},\\
&s_2 = a_{12} + a_{22},
\end{aligned} \tag{26}$$

we are able to eliminate the velocities $\dot{q}^1$ and $\dot{q}^2$ by the following relations

$$\dot{q}^1 = V\, f_1, \qquad \dot{q}^2 = V\, f_2, \tag{27}$$

where

$$V = \sqrt{\dot{q}_1^2 + \dot{q}_2^2}. \tag{28}$$

The choice of functions f_1 i f_2 gives the different form of the principle of the work and energy (6).

Furthermore we introduce the functions with parameter $z = z(t)$

$$f_1 = \frac{1 - z^2}{1 + z^2}, \qquad f_2 = \frac{2\,z}{1 + z^2}, \tag{29}$$

wherefrom the principle of the work and energy (6) has the form

$$\varphi_1 + \psi_1 \dot{V} + \rho_1 V \dot{z} = 0 \tag{30}$$

and integrand (9) gets the following form

$$\begin{aligned}
F &= 1 + \lambda(\varphi_1 + \psi_1 \dot{V} + \rho_1 V \dot{z}) +\\
&+ \dot{\sigma}_1^*(\dot{q}^1 - V f_1) + \dot{\sigma}_2^*(\dot{q}^2 - V f_2) = 0
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&\psi_1 = 2r_3(1 - z^4)z + r_6(1 - z^2)z^2 + r_5(1 + z^6),\\
&\rho_1 = 2d_{12}^*(1 - z^4) - 4r_4 z(1 - z^2) - 4(d_{12}^* - 2d_{21}^*)z^2\\
&\varphi_1 = r_1 + (2r_2 + r_1 z)z + (4r_2 - r_1 z)z^3 + (2r_2 - r_1 z)z^5\\
&r_1 = b_1^* + c_1^*, \quad r_2 = b_2^* + c_2^*,\\
&r_3 = d_{12}^* + d_{21}^*, \quad r_4 = d_{11}^* - d_{22}^*,\\
&r_5 = 1 + d_{11}^*, \quad r_6 = 3 - d_{11}^* + 4d_{22}^*.
\end{aligned} \tag{32}$$

Formulating Euler's equations for (31) in relation to q_1, q_2, V i z, we get (cf.(19))

$$\begin{aligned}
&\dot{\sigma}_1^* = 0 \;\rightarrow\; \sigma_1^* = C_1^* = const.,\\
&\dot{\sigma}_2^* = 0 \;\rightarrow\; \sigma_2^* = C_2^* = const.,\\
&\theta_2 + \dot{\lambda}\psi_1 + (\psi_1' - \rho_1)\dot{z}\lambda = 0,\\
&V(\theta_1 + \dot{\lambda}\rho_1) + \lambda[\dot{V}\left(\rho_1 - \psi_1'\right) + \varphi_1'] = 0,
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
&\theta_1 = -C_1^* \frac{4z}{(1 + z^2)^2} + C_2^* \frac{2(1 - z)}{1 + z^2},\\
&\theta_2 = C_1^* \frac{1 - z^2}{1 + z^2} + C_2^* \frac{2z}{1 + z^2},\\
&\psi_1' = 2r_6(1 + 2z^2)z + 2r_3(1 - 5z^4) + 6r_5 z^5,\\
&\phi_1' = 2r_2(1 + 6z^2 + 5z^4) + 2r_1(1 - 2z^2 - 3z^4)z.
\end{aligned} \tag{34}$$

The condition of transversality at the right end-point gets the form

$$[1 + \lambda\phi_1 - \theta_2 V]_{(t = t_1)} = 0, \tag{35}$$

and Euler's equations (33) have the first integral

$$1 + \lambda\varphi_1 - \theta_2 V = C, \quad C = 0. \tag{36}$$

Taking into consideration that V and z is not prescribed in the final position of the system, the end-conditions have the following form

$$[\frac{\partial F}{\partial \dot{V}}]_{(t = t_1)} = 0, \;\; [\frac{\partial F}{\partial \dot{z}}]_{(t = t_1)} = 0, \tag{37}$$

wherefrom we get

$$\lambda_{(t = t_1)} = \lambda_1 = 0. \tag{38}$$

Eliminating $\dot{V}$ by (30), from Euler's equation (cf. (33)) in relation to z we are able to eliminate $\dot{\lambda}$ from Euler's equation (cf. (36)) in relation to V, wherefrom we get (cf. (36))

$$V = \frac{\psi_1\phi_1' + \phi_1(\rho_1 - \psi_1')}{-\theta_1\phi_1\psi_1 + \theta_2[\psi_1\phi_1' + \phi_1(2\rho_1 - \psi_1')]}, \tag{39}$$

$$\lambda = \frac{\theta_2\rho_1 - \theta_1\psi_1}{\theta_1\varphi_1\psi_1 + \theta_2[-\psi_1\varphi_1' + \varphi_1(-2\rho_1 + \psi_1')]}. \tag{40}$$

In order to obtain the equations of motion of the system considered (cf. (32),(34)) the differential equations (27) can be written in the form

$$\frac{dq^1}{dz} = \chi f_1, \qquad \frac{dq^2}{dz} = \chi f_2. \tag{41}$$

where

$$\begin{aligned}
\chi &= -\frac{\vartheta_2 [\vartheta_2 \vartheta_3 \rho_1 + \psi_1 (\vartheta_3 \vartheta_2' - \vartheta_2 \vartheta_3')]}{\vartheta_3^3 \varphi_1} \\
\vartheta_1 &= \psi_1 \varphi_1' + \varphi_1 (2\rho_1 - \psi_1'), \quad \vartheta_2 = \vartheta_1 - \rho_1 \varphi_1, \\
\vartheta_1' &= \psi_1' \varphi_1' + \psi_1 \varphi_1'' + \varphi_1' (2\rho_1 - \psi_1') + \varphi_1 (2\rho_1' - \psi_1''), \\
\vartheta_2' &= \vartheta_1' - \rho_1' \varphi_1 - \rho_1 \varphi_1', \quad \vartheta_3 = \vartheta_1 \theta_2 - \theta_1 \varphi_1 \psi_1, \\
\vartheta_3' &= \vartheta_1' \theta_2 + \vartheta_1 \theta_2' - \theta_1' \varphi_1 \psi_1 - \theta_1 \varphi_1' \psi_1 - \theta_1 \varphi_1 \psi_1'.
\end{aligned} \tag{42}$$

5. SOLUTION

Suppose that the mechanical system considered has a velocity $V(t_0) = 0$ at the start position then equation (39) gives

$$z_0 = \alpha \pm \beta \tag{43}$$

where

$$\begin{aligned}
\alpha &= \frac{r_1 (1 + d_{22}^*) - d_{12}^* r_2}{d_{22}^* r_1 - r_5 r_2}, \\
\beta &= \sqrt{1 + \frac{(r_1 (1 + d_{22}^*) - d_{12}^* r_2)^2}{(d_{22}^* r_1 - r_5 r_2)^2}}.
\end{aligned} \tag{44}$$

Taking into account conditions (37) at the final position of the system considered, equation (40) gives (cf.(38))

$$C_1^* = K C_2^*, \tag{45}$$

where

$$K = \frac{r_5 (1 - z_1^2) + 2 d_{21}^* z_1}{2(1 + d_{22}^*) z_1 + d_{21}^* (1 - z_1^2)}. \tag{46}$$

Integration of differential equations (41) yields to general solutions

$$q^1 = \frac{1}{(C_2^*)^2} \Phi_1 + A_1, \quad q^2 = \frac{1}{(C_2^*)^2} \Phi_2 + A_2 \tag{47}$$

where

$$\begin{aligned}
&\Phi_1(z_1, z) = \int \chi_1 f_1 \, dz, \quad \Phi_2(z_1, z) = \int \chi_1 f_2 \, dz, \\
&A_i = q^{i0} - \frac{1}{(C_2^*)^2} \Phi_{i0}, \qquad q^{i0} = q^i(z_0), \\
&\Phi_{i0} = \Phi_i(z_1, z_0), \qquad i = 1,2,
\end{aligned} \tag{48}$$

in which (cf.(46))

$$\begin{aligned}
\chi_1 &= \frac{(1+z^2)^3 \vartheta_2 [\tilde{\vartheta}_2 \vartheta_2 - \tilde{\vartheta}_1 (1+z^2) \psi_1]}{\tilde{\psi}_1^3 \phi_1} \\
\tilde{\vartheta}_1 &= -2 p_1 \tilde{\phi}_1 \vartheta_2 + p_2 (1+z^2) \vartheta_2 \vartheta_1' + \tilde{\psi}_1 \vartheta_2', \\
\tilde{\vartheta}_2 &= p_2 (1+z^2)^2 \vartheta_1 \rho_1 - 2 p_1 \tilde{\rho}_1 (1+z^2) \psi_1 - 4 p_4 \phi_1 \psi_1^2, \\
\tilde{\rho}_1 &= \vartheta_1 + \rho_1 \phi_1, \quad \tilde{\phi}_1 = \psi_1 \phi_1' + \psi_1' \phi_1, \\
\tilde{\psi}_1 &= p_3 \vartheta_1 + 2 p_1 \phi_1 \psi_1', \\
p_1 &= -1 + 2Kz + z^2, \\
p_2 &= -2z + K(-1 + z^2), \\
p_3 &= K - Kz^4 + 2z(1 + z^2), \\
p_4 &= -K - 3z + 3Kz^2 + z^3.
\end{aligned} \tag{49}$$

Taking into account that at final position the following relations

$$\Delta(z_1) = \Delta q \Delta \Phi_2 - \Delta \Phi_1 = 0, \tag{50}$$

holds where (cf.(48))

$$\begin{aligned}
&\Delta \Phi_i = \Phi_{i1} - \Phi_{i0}, \quad i = 1,2, \\
&\Delta q = \frac{\Delta q^1}{\Delta q^2}, \quad \Delta q^i = q^i - q^{i0},
\end{aligned} \tag{51}$$

we get a value of the parameter z_1 at moment t_1 , and constant C_2^* (cf.(51))

$$C_2^* = \sqrt{\frac{\Delta \Phi_i}{\Delta q^i}}. \quad i = 1,2. \tag{52}$$

6. EXAMPLE

Let us consider the motion of the mechanical system which consists of three prismatic rigid bodies and moves in homogeneous field of gravity. The configuration of the system is defined by the set of coordinates $\bar{q}^1 = (\bar{q}^1, \bar{q}^2)$. System starts from position defined by coordinates $\bar{q}^1(t_0) = \bar{q}^{10}$ and $\bar{q}^2(t_0) = \bar{q}^{20}$ where it was at rest.

Final position is set by:

$$\bar{q}^1(t_1) = \bar{q}^{11} \text{ and}$$

$$\bar{q}^2(t_1) = \bar{q}^{21} .$$

The coefficient of Coulomb friction on the rough inclined side (at angle α to horizontal) of prism P_1 is μ_1. The coefficient of friction on rough vertical plane is μ_2 and the coefficient of friction on rough horisontal plane is μ_3 (fig 1.). Let m_1 denote the mass of the prism P_1, m_2 denote the mass of the prism P_2 and let m_3 denote the mass of the prism P_3 in a suitable system of unites. Prisms P_1 and P_2 are attached for the rope. Rope passes over drum without friction. The rotation of the drum is not resisted by friction.

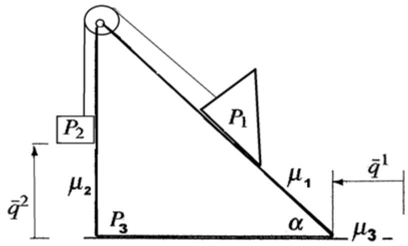

Figure.1 An example of the mechanical system

Taking into account that the differential equations of motion of the system considered (fig.1) have the form

$$a_{ij}\ddot{q}^j = \overline{Q}_i^{\mu} - \frac{\partial\overline{\Pi}}{\partial\overline{q}^i} + u_i \;, \quad i, j = 1,2, \tag{53}$$

where

$$Q_1^{\mu} = -\mu_3(Mg - M_1\ddot{\bar{q}}^2),$$

$$Q_2^{\mu} = -\mu_1 m_1 g\cos\alpha + (\mu_1 m_1 \sin\alpha - \mu_2 m_2)\ddot{\bar{q}}^2,$$

$$\frac{\partial\overline{\Pi}}{\partial\bar{q}^1} = 0, \quad \frac{\partial\overline{\Pi}}{\partial\bar{q}^2} = gM_1, \tag{54}$$

the relations (19) get the form

$$c_1^* = \frac{1}{\sqrt{a}}M_1, \quad c_1^* = -\frac{gs}{\sqrt{b}}M_1,$$

$$b_1^* = \frac{1}{\sqrt{a}}(gM\ \mu_3 + gm_1\mu_1\cos\alpha),$$

$$b_2^* = \frac{1}{\sqrt{b}}(gM\ \mu_3 + gsm_1\mu_1\cos\alpha),$$

$$d_{11}^* = \frac{1}{a}(-m_2\mu_2 + m_1\mu_1\sin\alpha - \mu_3 M_1),$$

$$d_{12}^* = \frac{1}{\sqrt{ab}}[-m_2\mu_2 + m_1\mu_1\sin\alpha - s\mu_3 M_1),$$

$$d_{21}^* = \frac{1}{\sqrt{ab}}[-\mu_3 M_1 + s(-m_2\mu_2 + m_1\mu_1\sin\alpha),$$

$$d_{22}^* = \frac{s}{b}[(-m_2\mu_2 + m_1\mu_1\sin\alpha) - \mu_3 M_1], \tag{55}$$

where

$$M = m_1 + m_2 + m_2,$$

$$M_1 = m_1\sin\alpha - m_2,$$

$$a = M + m_2 + m_1(1 + 2\cos\alpha),$$

$$b = M + (m_2 + m_1)s^2 + 2m_1 s\cos\alpha,$$

$$s = \frac{M + m_1\cos\alpha}{M - m_3 + m_1\cos\alpha}. \tag{56}$$

If $\alpha = \frac{\pi}{6}$, $m_1 = \frac{32}{13}$, $m_2 = m_1\sin\alpha$, $m_3 = \frac{m_1}{10}$, $\mu_1 = \frac{1}{20}$, $\mu_2 = \frac{\mu_1}{3}$, $\mu_3 = \frac{\mu_1}{5}$ then the relation gives the value of parameter $z(t)$ (at $t = t_0$) $z_0 = 0.4187046$. If $q^1(t_1) = 8.79$ and $q^2(t_1) = 4.1$ (at $t = t_1$) then relation (50) gives the value of the parametar $z_1 = 1.23232$.

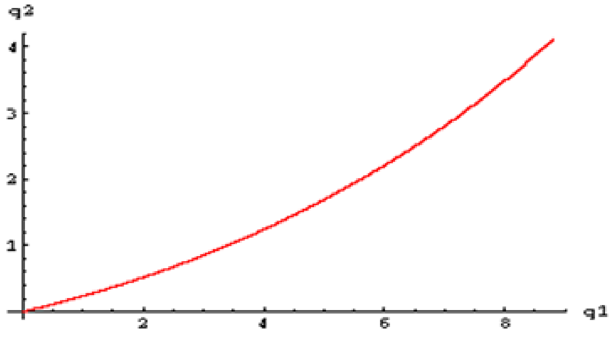

Figure.2 Brachistohrone of the mechanical system considered

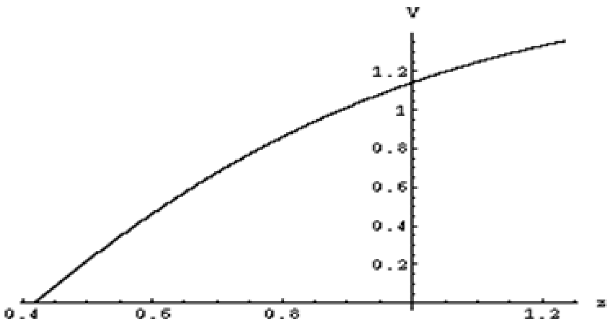

Figure.3 The graph of velocity V(z)

The graph in (Fig.2) is showing the brachistochone $q^2 = f(q^1)$ and graph in (Fig.3) is showing $V = V(z)$ (cf.(39)) of the system considered.

7. CONCLUSION

The paper considers a case of the brachistochronic motion of the mechanical system in the field of conservative forces, subject to the action of constrains with Coulomb friction. The constraint represents the modified form of the principle of work and energy (30) obtained in [6] and the new Euler's equations are formed. The complite analogy is made among solution obtained in the example considered, solution in [6] and the solution in relation to material point in [3].

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БРАХИСТОХРОНО КРЕТАЊЕ МЕХАНИЧКОГ СИСТЕМА СА РЕАЛНИМ ВЕЗАМА

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У овом раду разматрано је брахистохроно кретање у специјалном случају система са два степена слободе који се креће у пољу конзервативних сила под дејством веза са Кулоновим трењем. Избором функција генералисаних брзина извршена је параметризација диференцијалних једначина кретања. Формиран је нов математички модел који је коришћен за добијање брахистохроне и направљена је аналогија са математичким моделима који су изложени у радовима [3], [4] , [5] и [6].