Remarks on Perspective Simplices

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The planar figure of two triangles being in perspective position is associated with a theorem of Desargues, which is responsible for the fact that the coordinate set of the plane is a field. This well-known theorem of Desargues allows the obvious interpretation in space of a three-sided pyramid, which is intersected by two planes. The article is dedicated to generalizing this theorem of Desargues to perspective simplices in a projective n-space and their linear images in a subspace. Hereby one can find remarkable incidences and configurations. Starting point to this investigation is a planar figure of perspective quadrangles, where the second author discovered a remarkable coincidence figure. The proof of this incidence statement is based on an interpretation of the planar "Ebisui figure" as the central projection of a (projective) cross-polytope in 4-space. This principle can also be extended to higher dimensions.

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1. INTRODUCTION AND EBISUI'S THEOREM

Let π be a projective Desargues plane such that its coordinate field $\mathcal{F}$ has not the characteristic 2. As a consequence of these assumptions π can be immerged into a projective space of dimension $n \geq 3$ and there exist harmonic quadruplets of collinear points. (Plane π is a Fano plane, i.e. each quadrangle in π has a triangle of diagonal points; confer e.g. [1].) For such a plane π the following theorem holds, see also Figure 1.

Theorem 1 ("Ebisui's Theorem"): Let (A_i, B_i, C_i, D_i), $i = 1,2$, be two Z-perspective quadrangles of π . Then the intersection points

$$A_1B_1 \cap A_2B_2 =: E, \quad C_1D_1 \cap C_2D_2 =: \bar{E},$$
$$B_1C_1 \cap B_2C_2 =: F, \quad D_1A_1 \cap D_2A_2 =: \bar{F}, \qquad (1)$$
$$A_1C_2 \cap A_2C_1 =: G, \quad B_1D_2 \cap B_2D_1 =: \bar{G},$$

define three lines $E\bar{E}, F\bar{F}, G\bar{G}$ which are incident with a common point H.

(This theorem was communicated by the second author at the „Meeting of the Japan Society for Geometry and Graphics", Mai 12-13, 2007, Tokyo.)

A simple but lengthy proof of Theorem 1 might use elementary affine vector calculus in π assuming that none of the given and calculated points is an ideal point. Of course one will use an affine (or even projective) coordinate frame in π, which is suitably connected with the given point set $\{Z, A_i, ..., D_i\}$ to make calculation as simple as possible. But such an analytic proof does not show, where Theorem 1 belongs to and in which way it generalises the well-known Desargues' theorem of

perspective triangles. As the Theorem 1 deals only with incidences, it obviously belongs to projective geometry based on a rather general coordinate field $\mathcal{F}$, even it, at first, was stated as a theorem in a Euclidean plane.
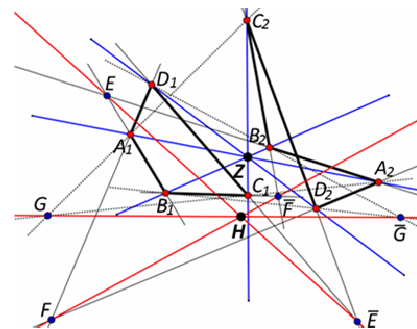

Figure 1. Two perspective quadrangles lead to an "opposite" point *H* of the perspectivity centre *Z*.

Note that the points $G, \bar{G}$ stem from "overcrossing" pairs of corresponding points of the two sets $\{A_1,..., D_1\}$ and $\{A_2,..., D_2\}$ and that the Desargues axes of the four pairs of triangles $(A_1B_1C_1, A_2B_2C_2)$, $(B_1C_1D_1, B_2C_2D_2)$, $(C_1D_1A_1, C_2D_2A_2)$, and $(D_1A_1B_1, D_2A_2B_2)$ form a complete quadrilateral with the six vertices $E\bar{E}, F\bar{F}, G\bar{G}$!

In the following we interpret the planar figure to Theorem 1 as an image of an object in space, similar to the classical interpretation of the (planar) Theorem of Desargues as linear image of a triangular pyramid, which is intersected by two planes. This allows a proof of Theorem 1 by simple reasoning instead of calculation and it gives a hint, how to generalize it to higher (and even lower) dimensions:

We interpret the planar "Ebisui figure" as the central projection of a cross-polytope in 4-space. Even so one at first might start with a regular cross-polytope in the Euclidean 4-space, the proof holds also for 4-spaces over any field $\mathcal{F}$ with char $\mathcal{F} \neq 2$. Using this idea allows to interpret any (classical) Desargues figure as

the central projection of a regular octahedron in the projective enclosed Euclidean 3-space. Similarly, as any quadrangle Q can be interpreted as central projection of a square Q', the quadrangle Q together with its 6 lines, i.e. the 'complete' quadrangle, can be interpreted as the 2-dimensional case of an Ebisui figure.

For dimensions $n > 4$ there exist incidence figures, too, but we end up with rather complicated configurations. Anyway, higher dimensional interpretations of perspective simplices and their connection to cross-polytopes still are possible and will result in incidence statements for those cross-polytopes.

2. DESARGUES' THEOREM AND CENTRAL AXONOMETRY

The well-known theorem of Desargues concerns two Z-perspective triangles and deals with incidences alone. If the place of action is a projective plane π such that Desargues' theorem is valid, then π is embeddable into a projective 3-space Π, and the Desargues theorem becomes obvious by the well-known interpretation of the figure as the linear image of an object in Π, namely of a 3-sided pyramid intersected by two planes, see Figure 2.
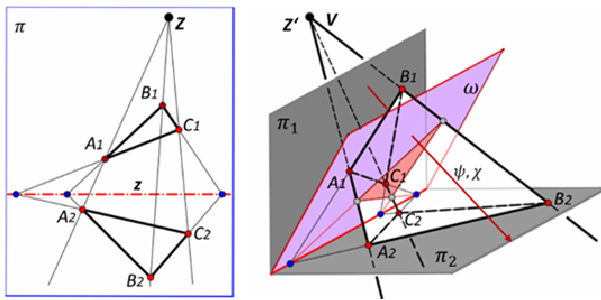
Figure 2. (a) general case of a (labelled) Desargues figure (left) (b) interpretation of (a) as image of an object in 3-space, which defines a perspectivity Ψ and a harmonic homology y.

If the Desargues axis z of a (labelled) Desargues figure (see Figure 2(a)) does not pass through the pers–pectivity centre Z, this figure defines a homology $\kappa:\pi \rightarrow \pi$ with centre Z, axis z and a pair (A_1, A_2) of corresponding points. The cross ratio $CR(A_2, A_1, Z, z) =: c \in \mathcal{F}$ is called the "characteristic cross ratio" of κ.

The Desargues figure shows an elation $\kappa:\pi \rightarrow \pi$, if $Z \in z$. In this case a characteristic cross ratio is not defined.

Remark: We distinguish the concepts "Desargues *figure*" and "Desargues *configuration*". The $(10_3, 10_3)$ - *configuration* of Desargues (Figure 2a) is a set of 10 points and 10 lines and no point (line) is distinguished from any other point (line). The concept characteristic cross-ratio does not make sense for such an "un-labelled" configuration. Taking another point of this configuration as center Z leads (in general) to another characteristic cross ratio value. The relation of all possible characteristic cross ratios to a Desargues configuration is described in [6].

In the 3-space Π the pre-image of the planar homology or elation κ is a *perspectivity* $\psi:\pi_1 \rightarrow \pi_2$ of a plane π_1 onto a plane π_2 with the pyramid's vertex V as

the perspectivity centre, see Figure 2(b). This perspectivity ψ can be embedded into *perspective collineations* $\chi:\Pi \rightarrow \Pi$ and there is a one-parametric set of such collineations, as we may choose any plane ω through $\pi_1 \cap \pi_2$ as fixed plane. Among these colline–ations χ, for any coordinate field $CR(A_2, A_1, Z, z) =: c \in \mathcal{F}$ of char $\mathcal{F} \neq 2$, besides the two singular ones with $\omega = \pi_1$ or $\omega = \pi_2$ as axis, there are two canonically distinguished regular ones: the *elation* χ_0 with the fixed plane $(\pi_1 \cap \pi_2) \vee V =: \omega$ and the *harmonic homology* χ_{-1} with an axis ω such that $CR(\pi_2, \pi_1, z \vee Z, \omega) = -1$. (We supposed π to be a Fano plane, therefore harmonic homologies in π and in Π are well defined regular perspective collineations.)

As we finally aim at the "overcrossings" occurring in Theorem 1 it seems to be natural to consider only the harmonic homology χ_{-1} as the distinguished collineation within the set of all perspective collineations belonging to ψ.

Let us at first consider a classical Desargues figure in a projective plane π which we at first embed into a projective enclosed affine 3-space Π_α. The basic figure of two Z-perspective triangles $\{Z; A_1, B_1, C_1; A_2, B_2, C_2)$ $\subset \pi$ can be interpreted as the fundamental figure of a central axonometry $\alpha: \Pi_\alpha \rightarrow \pi$.

Remark: „central axonometry" means a linear mapping of a space Π onto an image plane π based on a projective coordinate frame in Π and its linear image figure, which usually is called the "axonometric fundamental figure", (see e.g. [2], [3], [4] and [5]). Extensions to higher dimensions are obvious.
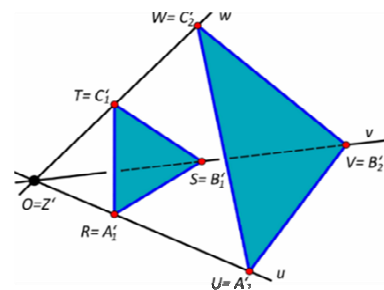
Figure 3. An axonometric fundamental figure connected with a Desargues figure.

Thereby $Z =: O^\alpha$ is the image of the origin O in space, furthermore we use A_1, B_1, C_1 as the images of "unit points" $R = A_1^{\alpha^{-1}}, S = B_1^{\alpha^{-1}}, T = C_1^{\alpha^{-1}}$, and interpret A_2, B_2, C_2 as the images of "ideal points" $U = A_2^{\alpha^{-1}}, V = B_2^{\alpha^{-1}}, W = C_2^{\alpha^{-1}}$ of the "coordinate axes" $u, v, w \subset \Pi$ through O, see Figure 3.

The "unit plane" $\varepsilon = RST$ intersects the "ideal plane" $v = UVW$ in the pre-image of the Desargues axis z of the two Z-perspective triangles. Thus the fixed plane ω of the harmonic homology χ_{-1} intersects u, v, w in the "negative unit points" R^-, S^- and T^-.

Remark: As we are free in choosing Π_α and its ideal elements, we could have started with a projective space Π and an axonometry such that R^-, S^-, T^- become the pre-images of A_2, B_2, C_2. Then the fixed plane ω of the above mentioned harmonic homology χ_{-1} automatically acts as the "ideal plane". Furthermore, if we assume that

$\mathcal{F}$ is a Euclidean field, then (Π, ω) allows an interpretation as a projective enclosed Euclidean space and R, S, T, R^-, S^-, T^- become the vertices of a regular octahedron with centre O. We collect this as:

Theorem 2: Let π be a projective plane over the field $\mathcal{F} \approx \mathbb{R}$ and A_1, B_1, C_1 and A_2, B_2, C_2 two Z-perspective triangles in π. Then $\{Z; A_1, B_1, C_1; A_2, B_2, C_2\} \subset \pi$ can always be interpreted as the central axonometric image of midpoint and vertices of a (Euclidean) regular octahedron.

3. THEOREM 1 AND CENTRAL AXONOMETRY

Let us now turn back to the planar configuration of two Z-perspective quadrangles $(A_i,..., D_i)$, i = 1, 2 as described in Theorem 1. Analog to Theorem 2 we now interpret this labelled configuration as the central axonometric image of an affine regular cross-polytope P_8 in the projectively extended four-dimensional affine space Π_α^4 over the field $\mathcal{F}$.

Again we interpret the two Z-perspective planar quadrangles at first as linear images of two O-perspective tetrahedra, each spanning a hyperplane Π_1 resp. $\Pi_2 \subset \Pi_\alpha^4$. We now embed the O-perspectivity $\psi : \Pi_1 \rightarrow \Pi_2$ into a harmonic homology $\chi : \Pi_\alpha^4 \subset \Pi_\alpha^4$ with centre O (the pre-image of Z) and with a certain hyperplane Ω as the fixed axis of χ. Let us choose Ω as the ideal hyperplane of Π_α^4 such that O is a proper point of Π_α^4. Then, by using the originals O and $A_1',...,D_1' \in \Pi_\alpha^4$ of Z, $A_1,..., D_1 \in \pi$ as origin and unit points of a coordinate frame, we receive very simple projective coordinates of the points of the entire configuration, namely:

$$A_i' = (1, \pm 1, 0, 0, 0)\,\mathcal{F}, \quad B_i' = (1, 0, \pm 1, 0, 0)\,\mathcal{F},$$
$$C_i' = (1, 0, 0, \pm 1, 0)\,\mathcal{F}, \quad D_i' = (1, 0, 0, 0, \pm 1)\,\mathcal{F}, \quad (2)$$
$$Z' = (1, 0, 0, 0, 0)\,\mathcal{F}, i = 1, 2.$$

Therewith we get the originals (c.f. (1)) E', $\bar{E}'$, F', $\bar{F}'$, G', $\bar{G}'$ of $E,...,\bar{G}$ almost without any calculation as:

$$E' = (0, 1, -1, 0, 0)\,\mathcal{F}, \quad \bar{E}' = (0, 0, 0, 1, -1)\,\mathcal{F},$$
$$F' = (0, 0, -1, 1, 0)\,\mathcal{F}, \quad \bar{F}' = (0, 1, 0, 0, -1)\,\mathcal{F}, \quad (3)$$
$$G' = (0, 1, 0, 1, 0)\,\mathcal{F}, \quad \bar{G}' = (0, 0, 1, 0, 1)\,\mathcal{F},$$

and finally

$$E'\bar{E}' = \{(\lambda(0, 1, -1, 0, 0) + \bar{\lambda}(0, 1, -1, 0, 0))\,\mathcal{F}\},$$
$$F'\bar{F}' = \{(\mu(0, 1, 0, 0, -1) + \bar{\mu}(0, 0, 1, -1, 0))\,\mathcal{F}\}, \quad (4)$$
$$G'\bar{G}' = \{(\nu(0, 1, 0, 1, 0) + \bar{\nu}(0, 0, 1, 0, 1))\,\mathcal{F}\},$$
$$(\lambda, \bar{\lambda}), (\mu, \bar{\mu}), (\nu, \bar{\nu}) \subset \mathcal{F}^2 \setminus (0, 0)$$

These three lines (4) intersect in a common point:

$$E'\bar{E}' \cap F'\bar{F}' \cap G'\bar{G}' =: H' = (0, 1, -1, 1, -1)\,\mathcal{F} \quad (5)$$

As the three lines $E', \bar{E}', F', \bar{F}', G', \bar{G}'$ of $\Omega \subset \Pi_\alpha^4$ indeed coincide with a common point H', any linear image of them must have the same property, i.e. the α-images $E\bar{E}, F\bar{F}, G\bar{G}$ of these lines are concurrent with a point $H = H'^\alpha$. Therewith we have proved

Theorem 3: Let be given a labelled planar "Ebisui figure" consisting of two Z-perspective quadrangles (A_i, B_i, C_i, D_i), i = 1,2 together with centre Z and the intersection points $E,...,\bar{G}$ according to Theorem 1.

- Then $\{Z; A_1,..., D_1; A_2,..., D_2\}$ can always be interpreted as the linear image (central axonometric image) of an affine regular cross-polytope P_8 with vertices $\{A'_1,..., D'_2\}$ and centre Z' in a four-dimensional projective extended affine space Π_α^4.

- Furthermore, the originals E', F', G' and $\bar{E}', \bar{F}', \bar{G}'$ of $E,...,\bar{G}$ form two perspective triangles in the ideal hyperplane Ω of Π_α^4 with a point H' as their Desargues centre, which maps onto the remarkable "Ebisui-point" H of the given Ebisui figure.

Remark 1: Obviously, if Π_α^4 is taken as a projective enclosed Euclidean 4-space, then P_8 can be interpreted as a Euclidean regular cross-polytope.

Remark 2: Note that central axonometry maps a point P' of the n-space to a point P of an image k-space or plane using the so-called *coordinate path* with respect to the axonometric base figure. The coordinate path uses the projective coordinates of P', therefore it is not necessary to distinguish between the original P' and their axonometric image P in terms of coordinates. In the following we therefore will omit to use different labelling for originals and their images, as it will come clear from the text what is meant.

4. COMPLETE QUADRANGLES AS THE TWO-DIMENSIONAL CASE: FANO'S FIGURE

Instead of perspective triangles in a projective Fano-plane π and its interpretation as (central) axonometric image of an octahedron let us now consider two line segments $s_1 := (A_1, B_1)$ $s_2 := (A_2, B_2) \subset \pi$ such that the endpoints of those segments form a quadrangle. These labelled segments define a unique perspectivity centre Z (see Fig. 4). Inter–pretation in space now degenerates to an interpretation in a projective enclosed affine (resp. Euclidean) plane π': $(A_1,..., B_2)$ are linear axonometric images of the vertices $(A'_1,..., B'_2)$ of a parallelogram (resp. a square); thereby point Z is the image of its centre Z'.

Again we embed the perspectivity $\psi : s'_1 \rightarrow s'_2$ into a harmonic homology $\chi : s'_1 \rightarrow s'_2$ and its axis z' passes through point $S' := A'_1B'_1 \cap A'_2B'_2$ and it contains the intersection G' of "overcrossing lines" $A'_1B'_2, A'_2B'_1$. The axonometry $\alpha : \pi' \rightarrow \pi$ is simply a projective colli–neation, we receive $CR(A_1B_1, A_2B_2, Z, z)$, what is a tri–vial property of complete quadrangles in a Fano plane π.

As a trivial result we state:

Theorem 4: A labelled complete quadrangle in a projective Fano plane allows an interpretation as the two-dimensional (degenerated) case of a general Desargues figure in the sense of H. Ebisui.
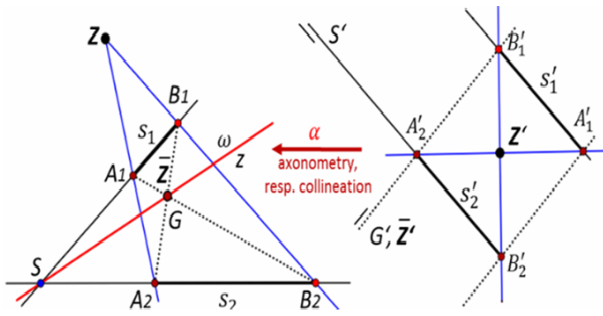
Figure 4. Two generally positioned labelled segments spanning a plane are trivially in perspective position and they give rise to a FANO figure as the two-dimensional case of Ebisui's construction.

5. EBISUI-POINTS OF PERSPECTIVE TETRAHEDRA AND COMPLETE QUADRANGLES

Already the four-dimensional case has to consider two sub-cases: the one described in Theorem 3 and one, where we start with a pair of perspective skew quadrangles in a 3-space Π^3. Also this figure is a proper axonometric basis figure for an axonometry $\alpha : \Pi^4_\alpha \to \Pi^3$ and therefore we can use the same calculation as for Theorem 3 receiving the same point $H' \in \Pi^4_\alpha$ as result.

Again we state as a:

Theorem 5: Let two labelled skew quadrangles in a 3-dimensional projective Fano space be given in perspective position. Then the three lines $E\overline{E}, F\overline{F}, G\overline{G}$ defined according to Theorem 1 have a common point H.

Remark 1: Two labelled tetrahedra in perspective position or two planar complete quadrangles in perspective position give rise to three pairs of edge quadrilaterals and each pair leads to an Ebisui point H_k, $k = 1,2,3$. This combinatorial approach, which we also want to apply to higher dimensions, makes a *re-labelling of the quadrangles* necessary:

Table 1. (Axonometric) projective coordinates of the Ebisui-points H^k to perspective tetrahedra.

Case 1: quadrangle (A_1, A_2, A_3, A_4)	Case 2: quadrangle (A_1, A_2, A_3, A_4)	Case 3: quadrangle (A_1, A_2, A_3, A_4)
$D_{22} = (0,1,-1,0,0)\,\mathcal{F}$	$D_{22} = (0,1,-1,0,0)\,\mathcal{F}$	$D_{24} = (0,0,1,0,-1)\,\mathcal{F}$
$D_{24} = (0,0,0,1,-1)\,\mathcal{F}$	$D_{24} = (0,0,0,1,-1)\,\mathcal{F}$	$D_{22} = (0,1,0,-1,0)\,\mathcal{F}$
$D_{22} = (0,1,0,-1,0)\,\mathcal{F}$	$D_{24} = (0,0,1,0,-1)\,\mathcal{F}$	$D_{22} = (0,0,1,-1,0)\,\mathcal{F}$
$D_{24} = (0,0,1,0,-1)\,\mathcal{F}$	$D_{22} = (0,1,0,-1,0)\,\mathcal{F}$	$D_{24} = (0,1,0,0,-1)\,\mathcal{F}$
$O_{22} = (0,1,0,1,0)\,\mathcal{F}$	$O_{24} = (0,1,0,0,1)\,\mathcal{F}$	$O_{22} = (0,1,1,0,0)\,\mathcal{F}$
$O_{24} = (0,1,0,1,0)\,\mathcal{F}$	$O_{22} = (0,0,1,1,0)\,\mathcal{F}$	$O_{24} = (0,0,0,1,1)\,\mathcal{F}$
$H^2 = (0,1,-1,1,-1)\,\mathcal{F}$	$H^2 = (0,1,-1,-1,1)\,\mathcal{F}$	$H^2 = (0,1,1,1,1)\,\mathcal{F}$
$Z = (1,0,0,0,0)\,\mathcal{F}, G = O_{13}O_{24} \cap O_{14}O_{24} \cap O_{12}O_{34} = (0,1,1,1,1)\,\mathcal{F}$		

In the following we label the points of the first quadrangle with A_i, $i = 1,.., 4$ and those of the second quadrangle with B_i, $i = 1,.., 4$ The "direct Desargues

points" mean $A_iA_j \cap B_iB_j =: D_{ij}$, while Desargues points stemming from "overcrossings" are labelled by $O_{ij} := A_iB_j \cap B_iA_j$. We will also use the "ideal points" C_i of lines ZA_i and being harmonic to Z with respect to the pairs (A_i, B_i). The points C_i form a simplex (i.e. a tetrahedron) in the ideal hyperplane, i.e. a 3-space and have $(0,\delta_{1i},\delta_{2i},\delta_{3i},\delta_{4i})F$ as projective coordinates.

We list the three cases with the corresponding figures and coordinates below (Table 1, Figures 5 and 6):

From Table 1 we read off that there occur only six different points D_{ij}, and they are the vertices of a complete quadrilateral, the sides of which are the Desargues axes of the four pairs of partial triangles of the given pair of quadrangles resp. tetrahedra.

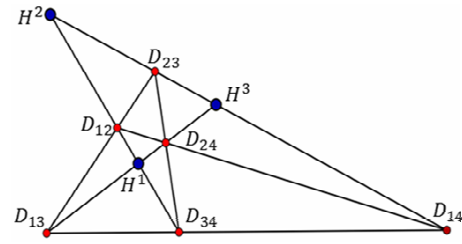The three Ebisui-points H^k are the diagonal points of the above mentioned quadrilateral (Figure 5).

Figure 5. Configuration of intersection points D_{ij} of corres–ponding sides of perspective tetrahedra and their three Ebisui points H^k.

In addition we can state:

Theorem 6: Given two tetrahedra or two planar quadrangles in perspective positions, then the three lines $O_{13}O_{24}, O_{14}O_{24}, O_{12}O_{34}$ stemming from "over-crossings" meet in a common point $G = (0,1,1,1,1)\,\mathcal{F}$. The three Ebisui-points H^k together with G have (axonometric) projective coordinates, which can be interpreted as vertices of a regular tetrahedron in the ideal hyperplane of a projectively enclosed affine 4-space Π^4_α.

We show the mutual incidences described in Theorem 6 in an axonometric view of that ideal hyperplane of Π^4_α in Figure 6.

Remark 2: Among other incidences we find that the lines $D_{12}O_{34}, D_{13}O_{24}$ and $D_{14}O_{23}$ have a common point $F^1 = (0,-1,1,1,1)F$ and similarly

$$D_{12}O_{34} \cap D_{23}O_{14} \cap D_{24}O_{13} = \mathcal{F}^2 = (0,1,-1,1,1)\,\mathcal{F},$$

$$D_{13}O_{24} \cap D_{23}O_{14} \cap D_{34}O_{12} = \mathcal{F}^3 = (0,1,1,-1,1)\,\mathcal{F},$$

$$D_{14}O_{23} \cap D_{24}O_{13} \cap D_{34}O_{12} = \mathcal{F}^4 = (0,1,1,1,-1)\,\mathcal{F}.$$

The simplices ($\mathcal{F}^1,..., \mathcal{F}^4$) and ($C^1,..., C^4$) are G-perspective, while ($\mathcal{F}^1,..., \mathcal{F}^4$) and ($G,H^1,H^2,H^3$) are not only C_1-perspective, but also C_i-perspective, see Figure 6.

Remark 3: The points O_{ij} can be interpreted as vertices of an octahedron with centre G. We also recognize that, within the ideal hyperplane of Π^4_α, i.e. a 3-space Π^3_ω, this octahedron defines a Desargues configuration and e.g. a perspectivity ψ_3 with centre G mapping the triangle $(O_{12}O_{13}O_{14})$ onto the triangle $(O_{23}O_{34}O_{24})$ and again ψ_3 can be embedded into a harmonic homology χ_3 with axis plane spanned by the complete quadrilateral $\{D_{ij}\}$, see Fig. 5. The three collinear points D_{34}, D_{24}, D_{23}

span the Desargues axis of the perspectivity ψ_3, i.e. the lines $O_{12}O_{13}$ and $O_{34}O_{24}$ meet at D_{23}, and so on. But there are altogether four possibilities of such perspectivities resp. homologies and each side of the complete quadrilateral $\{D_{ij}\}$ acts as Desargues axis of each of these perspectivities.
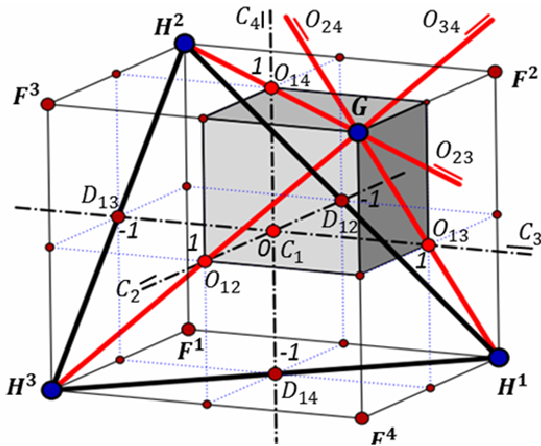
Figure 6. Configuration of intersection points O_{ij} of "over–crossing" connecting sides of perspective tetrahedra and their three Ebisui points H^k.

Our cross-polytope P_8, which realises a perspectivity ψ_4 between two opposite face tetrahedra and the canonically defined harmonic homology χ_4 to ψ_4, induces an octahedron P_6 in the (ideal) axis hyperplane of χ_4. This octahedron again delivers the classical situation of a Desargues figure as treated in Chapter 2 with the two-dimensional case of a complete quadrangle in the axis plane of χ_3, see Chapter 4. In the following we shall see that this hierarchical structure also holds for higher-dimensional cases.

6. HIGHER-DIMENSIONAL CASES

The idea of interpreting the planar figures as axonometric image of a square, an octahedron or a cross-polytope P_8 suggests to consider pairs of closed polygons with p vertices in perspective position in a given classical projective k-space Π^k and interpret them as the central axonometric image of a cross-polytope P_{2p} in an n-dimensional affine (resp. Euclidean) space Π_α^p with an "ideal hyperplane" Π_ω^{p-1} as projective enclosure. The polygons $(A_1,..., A_p)$ and $(B_1,..., B_p)$ together with centre Z give reason to $\binom{p}{4}$ quadrangles and each of them lead to three Ebisui-points and one point G. There occur $\binom{p}{2}$ direct Desargues points $D_{ij} = A_iA_j \cap B_iB_j$ and $\binom{p}{2}$ points O_{ij} stemming from "overcrossing" lines A_iB_j, A_jB_i. Interpreting the points of the two polygons as vertices of P_{2p} with centre Z (- we use the same symbols for points in Π^k and in Π_α^p -) the points D_{ij} and O_{ij} as well as the Ebisui-points and the points of type G span the ideal hyperplane Π_ω^{p-1}. The p intersections of $\Pi_\omega^{p-1} \cap ZA_i =: C_i$ can be used to

define a coordinate frame in Π_ω^{p-1}. The three hyper–planes Π_ω^{p-1}, Π_A^{p-1}, Π_B^{p-1}, spanned by $\{C_i\}$. $\{A_i\}$ resp. $\{B_i\}$ intersect in a space Π_ω^{p-2} spanned by the set $\{D_{ij}\}$, see Figure 7.

In the following we present only the case p = 5 as an example. It illustrates already the occuring incidences and configurations also for arbitrary cases of p:

Let two Z-perspective 5-gons $\{A_i\}$,$\{B_i \neq A_i\}$ be given in a projective k-space ($2 \leq k \leq 5$), then, after interpreting this set of 11 points (which we assume to be different) as an axonometric fundamental figure of an axonometry $\alpha:\Pi^5 \to \Pi^k$, the originals Z resp. A_i resp. B_i the homogenous coordinates $(1,0,0,0,0)$ $\mathcal{F}$ resp. $(1,\delta_{1i},..., \delta_{5i})$ and $(1,-\delta_{1i}, -\delta_{2i}, -\delta_{3i}, -\delta_{4i}, -\delta_{5i},)$ form a 5-cross-polytope P_{10}. By this we get 10 "direct" and 10 "overcrossing" Desargues points as follows (Table 2):
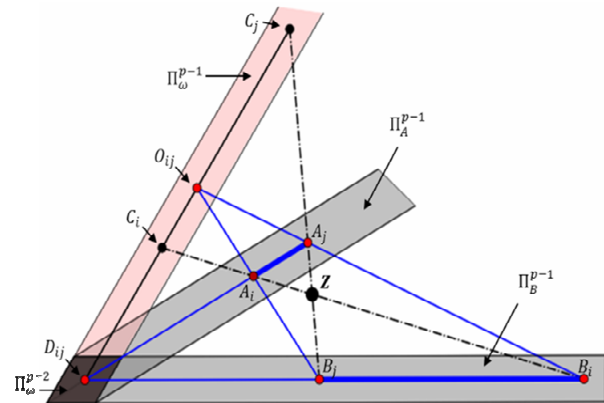
Figure 7. Symbolic visualisation of perspective p-simplexes and their "direct" and "overcrossing" Desargues points D_{ij} and O_{ij}.

Table 2. : (Axonometric) projective coordinates of the direct and overcrossing Desargues points to perspective 5-gons

$D_{12} = A_1A_2 \cap B_1B_2 =$ $= (0,1,-1,0,0,0)$ $\mathcal{F}$	$O_{12} = A_1B_2 \cap B_1A_2 =$ $= (0,1,1,0,0,0)$ $\mathcal{F}$
$D_{22} = A_2A_2 \cap B_2B_2 =$ $= (0,1,0,-1,0,0)$ $\mathcal{F}$	$O_{22} = A_2B_2 \cap B_2A_2 =$ $= (0,1,0,1,0,0)$ $\mathcal{F}$
$D_{14} = A_1A_4 \cap B_1B_4 =$ $= (0,1,0,0,-1,0)$ $\mathcal{F}$	$O_{14} = A_1B_4 \cap B_1A_4 =$ $= (0,1,0,0,1,0)$ $\mathcal{F}$
$D_{15} = A_1A_5 \cap B_1B_5 =$ $= (0,1,0,0,0,-1)$ $\mathcal{F}$	$O_{15} = A_1B_5 \cap B_1A_5 =$ $= (0,1,0,0,0,1)$ $\mathcal{F}$
$D_{22} = A_2A_2 \cap B_2B_2 =$ $= (0, 0, 1,-1,0,0,)$ $\mathcal{F}$	$O_{22} = A_2B_2 \cap B_2A_2 =$ $= (0,1,1,0,0,0)$ $\mathcal{F}$
$D_{24} = A_2A_4 \cap B_2B_4 =$ $= (0,1,0,0,-1,0)$ $\mathcal{F}$	$O_{24} = A_2B_4 \cap B_4A_2 =$ $= (0,0,1,0,1,0)$ $\mathcal{F}$
$D_{23} = A_2A_3 \cap B_2B_3 =$ $= (0,0,1,0,0,-1)$ $\mathcal{F}$	$O_{23} = A_2B_3 \cap B_3A_2 =$ $= (0,0,1,0,0,1)$ $\mathcal{F}$
$D_{24} = A_2A_4 \cap B_2B_4 =$ $= (0,0,0,1,-1,0)$ $\mathcal{F}$	$O_{24} = A_2B_4 \cap B_4A_2 =$ $= (0,0,0,1,1,0)$ $\mathcal{F}$
$D_{25} = A_2A_5 \cap B_2B_5 =$ $= (0,0,0,1,0,-1)$ $\mathcal{F}$	$O_{25} = A_2B_5 \cap B_5A_2 =$ $= (0,0,0,1,0,1)$ $\mathcal{F}$
$D_{45} = A_4A_5 \cap B_4B_5 =$ $= (0,0,0,0,1,-1)$ $\mathcal{F}$	$O_{45} = A_4B_5 \cap B_5A_4 =$ $= (0,0,0,0,1,1)$ $\mathcal{F}$

From Table 2 follows that the ten triplets of direct Desargues points (D_{ij}, D_{jk}, D_{ik}) are collinear with lines d_{ijk} and that all those points D_{ij} span a 3-space $\Pi_\omega^3 \subset \Pi_\omega^4 \subset \Pi_\omega^5$. As expected we find that the 10 points D_{ij} and 10 lines d_{ijk} form a Desargues configuration in Π_ω^3 with five planes Q_ω^m, $m = 1, ..., 5$, () ($m \neq i \neq j \neq k \neq$ l), containing 4 lines d_{ijk} and six points D_{ij}. These five planes correlate to the five Z-perspective pairs of partial quadrangles $Q_A^m = A_iA_jA_kA_l$, $Q_A^m = B_iB_jB_kB_l$ of the given perspective 5-gons, and each of the pairs $\left(Q_A^m, Q_B^m\right)$ leads to a triplet of Ebisui points $\left(H_1^m, H_2^m, H_3^m\right)$ in the corresponding plane $Q_\omega^3 \subset \Pi_\omega^3$ and a point $G^m \in \Pi_\omega^4$. We list the coordinates in Table 3:

Table 3. : (Axonometric) projective coordinates of the direct and overcrossing Desargues points to perspective 5-gons

$\left(Q_A^3, Q_B^3\right)$	$H_2^3 = \begin{pmatrix}0\\1\\-1\\1\\-1\\0\end{pmatrix}\mathcal{F}$	$H_2^3 = \begin{pmatrix}0\\1\\-1\\-1\\1\\0\end{pmatrix}\mathcal{F}$	$H_2^3 = \begin{pmatrix}0\\1\\1\\-1\\-1\\0\end{pmatrix}\mathcal{F}$	$G^3 = \begin{pmatrix}0\\1\\1\\1\\1\\0\end{pmatrix}\mathcal{F}$
$\left(Q_A^2, Q_B^2\right)$	$H_2^4 = \begin{pmatrix}0\\1\\-1\\0\\1\\-1\end{pmatrix}\mathcal{F}$	$H_2^4 = \begin{pmatrix}0\\1\\-1\\-1\\0\\1\end{pmatrix}\mathcal{F}$	$H_2^4 = \begin{pmatrix}0\\1\\1\\-1\\0\\-1\end{pmatrix}\mathcal{F}$	$G^4 = \begin{pmatrix}0\\1\\1\\1\\0\\1\end{pmatrix}\mathcal{F}$
$\left(Q_A^4, Q_B^4\right)$	$H_2^2 = \begin{pmatrix}0\\1\\-1\\0\\1\\-1\end{pmatrix}\mathcal{F}$	$H_2^2 = \begin{pmatrix}0\\1\\-1\\0\\-1\\1\end{pmatrix}\mathcal{F}$	$H_2^2 = \begin{pmatrix}0\\1\\1\\-1\\-1\\0\end{pmatrix}\mathcal{F}$	$G^2 = \begin{pmatrix}0\\1\\1\\0\\1\\1\end{pmatrix}\mathcal{F}$
$\left(Q_A^2, Q_B^2\right)$	$H_2^2 = \begin{pmatrix}0\\1\\0\\-1\\1\\-1\end{pmatrix}\mathcal{F}$	$H_2^2 = \begin{pmatrix}0\\1\\0\\-1\\-1\\1\end{pmatrix}\mathcal{F}$	$H_2^2 = \begin{pmatrix}0\\1\\0\\1\\-1\\-1\end{pmatrix}\mathcal{F}$	$G^2 = \begin{pmatrix}0\\1\\0\\1\\1\\1\end{pmatrix}\mathcal{F}$
$\left(Q_A^1, Q_B^1\right)$	$H_1^2 = \begin{pmatrix}0\\0\\1\\-1\\1\\-1\end{pmatrix}\mathcal{F}$	$H_1^2 = \begin{pmatrix}0\\0\\1\\-1\\-1\\1\end{pmatrix}\mathcal{F}$	$H_1^2 = \begin{pmatrix}0\\0\\1\\1\\-1\\-1\end{pmatrix}\mathcal{F}$	$G^1 = \begin{pmatrix}0\\0\\1\\1\\1\\1\end{pmatrix}\mathcal{F}$

From Tables 2 and 3 we read off that the ten lines G^mG^l pass through D_{mi} and that e.g. H^5, O_{12}, O_{34}, G^5 are collinear and harmonic, and we notice that e.g. the pair

of Ebisui triangles $\left(H_1^5H_2^5H_3^5\right)$ and $\left(H_1^2H_2^2H_3^2\right)$ is D_{34}-perspective. The "ideal points" $C_i \in \Pi_\omega^4$ of the five lines A_iB_i are harmonic to $Z = (1,0,..., 0)F$ with respect to (A_iB_i) and they are therefore well defined also in the axonometric image space Π^k, $2 \leq k < 5$.

The five pairs of points (C_i, G^i) are in perspective position with the "unit point" $G = (0,1,1,1,1,1)\,\mathcal{F}$ $\in \Pi_\omega^4$. The quadruplets $(D_{12}, D_{13}, D_{14}, D_{15})$ and $(O_{12}, O_{13}, O_{14}, O_{15})$ are C_I-perspective and form a 4-cross-polytope P_8 with centre C_I and "ideal points" (C_2, C_3, C_4, C_5) as expected.

Obviously there occur all together five such 4-crosspolytopes and the points C_i are their centres. The tetrahedron (C_2, C_3, C_4, C_5) contains the octahedron $(O_{23}, O_{24}, O_{34}, O_{35}, O_{45})$ as a cross-polytope P_6 with centre G^1, see Figure 8.
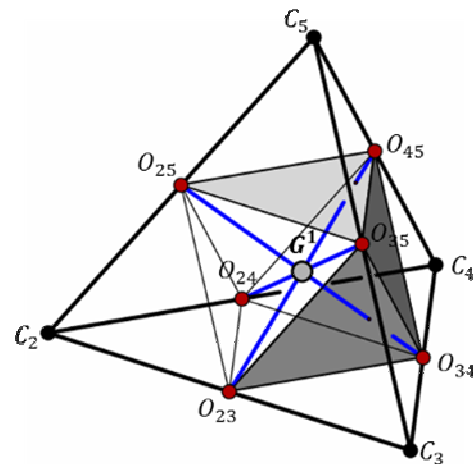
Figure 8. One of the five partial octahedra within the set of 10 "overcrossing" Desargues points O_{ij}.

These statements concerning incidences and relations occurring in a special-dimensional case should give sufficient insight into the combinatorial and analytic methods to treat also arbitrary dimensional cases. The occurrence of Desargues configurations and their higher dimensional analogues shall be mentioned in more detail in the next chapter.

7. GENERALIZED DESARGUES CONFIGURATIONS AND CROSS-POLYTOPES

In the former Chapters we found complete quadri–laterals (resp. quadrangles) and 3D-Desargues configu–ration connected with cross-polytopes. For such cross-polytopes one can present a list of facets in Table 4 (at the end of text), see also [7].

Each face-triangle or face-tetrahedron, together with "ideal" points and lines, gives rise to a complete face-quadrilateral resp. a Desargues configuration in the projective extended 3-space spanned by the face-tetrahedron.

Similarly, a k-face-simplex, together with its ideal elements, defines what might be called a "i- Desargues configuration" generalizing the standard case to higher dimensions. Also here the system of incidences shall be shown in a Table 5 (also at the end of text).

Table 4. Numbers of *k*-facets of a cross-polytope P₂ₙ

dim	name	points	lines	2-facets	3-facets	4-facets	5-facets	6-facets	$n-1$
1	segment	2	1						
2	quadrangle	4	4	1					
3	octahedron	6	12	8	1				
4	16-cell	8	24	32	16	1			
5	Pentacross	10	40	80	80	32	1		
6	Hexacross	12	60	160	240	192	64	1	
7	Heptacross	14	84	280	560	672	448	128	1
⋮	⋮	⋮	⋮	⋮					
n	n-Cross	points $2n$	lines $4\binom{n}{3}$	planes $8\binom{n}{4}$	k-facets: $2^{k+1}\binom{n}{k+1},\,k<n$				$(n$-1$)$-facets: 2^n

Table 5. Numbers of incident subspaces within *n*-dimensional generalized Desargues configurations

dim	incident with a	points	lines	planes	3-spaces	4-spaces	5-spaces	6-spaces	
2	point	1	2	1					
	line	3	1	1					
	plane	6	4	1					
3	point	1	3	3	1				
	line	3	1	2	1				
	plane	6	4	1	1				
	3-space	10	10	5	1				
4	point	1	4	6	4	1			
	line	3	1	3	3	1			
	plane	6	4	1	2	1			
	3-space	10	10	5	1	1			
	4-space	15	20	15	6	1			
5	point	1	5	10	10	5	1		
	line	3	1	4	6	4	1		
	plane	6	4	1	3	3	1		
	3-space	10	10	5	1	2	1		
	4-space	15	20	15	6	1	1		
	5-space	21	35	35	21	7	1		
6	point	1	6	15	20	15	6	1	
	line	3	1	5	10	10	5	1	
	plane	6	4	1	4	6	4	1	
	3-space	10	10	5	1	3	3	1	
	4-space	15	20	15	6	1	2	1	
	5-space	21	35	35	21	7	1	1	
	6-space	28	56	70	56	28	8	1	
n	point	1	$\binom{n}{k},\,k\ldots\dim subspaces$						
	line	3	1	$\binom{n-1}{k}$					
	plane	6	4	1	$\binom{n-2}{k}$				
	3-space	10	10	5	1	$\binom{n-3}{k}$			
	⋮	$\binom{p+2}{k},\,p\ldots\dim subspace\ k\ldots\dim subspaces$			1	$\binom{n-p}{k},\,p\ldots\dim subspace\ k\ldots\dim subspaces$			
	$(n$-1$)$-space	$\binom{n+1}{k}$					1	1	
	n-space	$\binom{n+2}{k}$						1	

8. CONCLUSION

Starting from the discovery of a "remarkable point" occurring at perspective quadrangles in the projective enclosed Euclidean plane by the second author we could now generalize this fact to projective spaces over any coordinate field F with char F ≠ 2 and any dimension n ≥ 2.

The key tool is the interpretation of the given pair of perspective p-gons as axonometric fundamental figure of a central axonometry mapping an affine cross-polytope of a p-dimensional projective enclosed affine space to the given pair of p-gons in the k-space spanned by them. As the axonometric coordinate path in the image k-space uses the same projective coordinates as are used in the original p-space, the calculation of occurring incidences acts on Zeroes and Ones alone.

REFERENCES

[1] Brauner, H.: Geometry of Projective Spaces I & II (Geometrie Projektiver Räume I,II). BI Wissenschaftsverlag, Mannheim/Wien/Zürich 1976, 3-411-01512-8 and 3-411-01513-6 Hochschultaschenbuch

[2] Havlicek, H: On the matrices of central linear mappings, Math. Bohem. 121 (1996), 151-156.

[3] Stachel, H.: More-dimensional Axonometry (Mehrdimensionale Axonometrie), in: N.K. Stephanidis (ed.), Proceedings of the Congress of Geometry, Thessaloniki 1987, 159-168.

[4] Stachel, H.,: On Arne Dür's equation concerning central axonometries, J. Geom. Graphics 8 (2004), 215-224.

[5] Stachel, H., Szabó, J. and Vogel, H.: A Theorem on Central Axonometry (Ein Satz über die Zentralaxonometrie), Sb. österr. Akad. Wiss., math.-naturw. Kl., Abt.II 203 (1994), 3-11.

[6] Weiss, G., Manhart, F.: About some mappings defined by a classical Desargues configuration. Journal for Geometry and Graphics vol. 16 (2012), Nr. 2, p. 1-12

[7] https://en.wikipedia.org/wiki/Cross-polytope (retri–eved April 18, 2016).

ЗАПАЖАЊА О ПЕРСПЕКТИВНИМ СИМПЛЕКСИМА

Г. Вајс, Х. Ебисуи

Два троугла једне равни која су у перспективном положају се повезују са Дезарговом теоремом која је заслужна за чињеницу да скуп координата у равни представља поље. Ова позната теорема омогућава очигледно тумачење тростране пирамиде у простору коју секу две равни.

Овај чланак је посвећен генерализацији Дезаргове теореме на перспективне симплексе у пројективном н-простору и њихових линеарних ликова у потпростору. На овај начин могу се открити значајне инциденције и конфигурације. Полазна тачка у овом истраживању је равни лик перспективних четвороуглова у коме је други аутор овог рада открио изузетно неочекивани лик. Доказ за ово изузетно тврђење се заснива на тумачењу равног лика бога Ебису као централне пројекције (пројективног) октаедра у 4Д простору. Овај принцип се може проширити и на већи број димензија.