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Limit Elastic Speeds of Functionally Graded Annular Disks

Based on Variational principle, the limit elastic angular speed of rotating disk made of functionally graded material is reported. Assuming a series approximation following Galerkin's principle, the solution of the governing equation is obtained based upon von Mises failure criterion. The elasticity modulus, density and yield stress are assumed to vary according to power law with grading index in the range of -3.0 to 3.0. At grading index, n = 0.0, the disk assumes isotropic material behavior. The investigation reports the variation of limit elastic speed with grading parameter for a different ratio aspect of the annular disk and establishes the existence of optimum grading index at each ratio aspect. The location of yield initiation is also reported in each case and is observed to play a significant role in optimizing limit elastic speed. Further, the displacement, strain and stress states of the disks at limit elastic speed is also reported. The results are validated with benchmarks for the appropriate system parameter values. Due to Variational nature of the solution and ease of handling the non-linear failure criterion, the solution methodology is observed to be stable, simple and robust.

Keywords: *Limit elastic speed, annulus, von-Mises criterion, FGM*

1. INTRODUCTION AND LITERATURE REVIEW

Ascertaining stress and displacement state in structures under mechanical loading is a vital design prerequisite. In the genre of axisymmetric structures, the basic problem is defined in terms of radial displacement field. These find vast applications in mechanical, automotive and aerospace industries in the form of thick and thin walled cylinders, flywheels, shrink fits, rotors and impellers, data storage devices, gears and pulleys to name a few [1, 2]. Stress and displacement depends greatly on the geometry, material and boundary conditions of the disk. Of these, effect of material composition on disk performance seizes maximum interest, primarily due to the recent advent of new materials and secondarily, selection of disk material, governed by the functional constraints, offers scope of optimizing the performance. In this context, Functionally Graded Materials or FGM finds a huge attention in recent researches. In FGM, properties vary continuously along a direction due to varying compositions thus optimizing the stress distribution. In axisymmetric structures, radial variation of material properties is addressed using one of the several distribution functions such as exponential, power law or Mori-Tanaka scheme. FGMs find huge application in the design and fabrication of engineering structures and subsequently there appears sufficient reports to gain insight into the stress and displacement behavior of FG structures.

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Till date, a host of literature have been published reporting the displacement and stress state of solid and annular disks and find ready reference in standard texts [3]. An early work, [4] presented finite difference solution of rotating disk of general profile and reported stresses due to prescribed bore displacements based on Southwell stress functions. Elastic stresses for rotating disk of varying geometry made of isotropic material possessing non-linear stress-strain relationship (Ramberg-Osgood stress-strain relations) has been reported by [5], using applied dynamic relaxation technique. In [6], a comparison of the solution of rotating solid and annular disks made of fibre-reinforced FGM is obtained using FEM with direct integration of coupled governing differential equation. The effect of material gradation on average stresses and displacement of the disk was reported. Based on Tresca's yield criterion and its associated flow rule, hyper geometric differential equations are solved in [7] to obtain stress distribution in rotating parabolic annular disks of linear strain hardening material behaviour. Exact solution of elastic stress up to the point of yielding is presented by [8], where in the analytical solution for elastic deformation of disks with parabolic thickness variation under pressurized boundary conditions is reported and the effect of geometry parameters on the location of yield initiation is established. Using Whittaker's functions, unified analytical solution in terms of elastic deformation for the exponentially graded rotating annular disks under clamped as well as free boundaries is reported in [9]. In [10], the derivation of a semi-analytical solution for rotating functionally graded disks modelled with radial virtual sub-domains having power-law distribution of thermo-mechanical properties is reported. Elastic stresses arising due to thermal load in rotating-converging conical disks with radial density variation is reported in [11] using hyper-geometric differential equation for the displacement field in disks. In another work, [12], rotating disk behaviour of disks having varying thickness with radial density variation under thermal load is presented. In [13], infinite power series based approximations for nonlinear equations using ADM and HPM is presented to obtain distributions of stresses and displacement in rotating annular elastic disks of varying geometry and densities. Elastic solutions for variable thickness rotating disks made of functionally graded material having power-law variations in geometry and material properties is reported in [14] wherein the effects of the material grading index and the geometry of the disk on the stresses and displacements has been investigated. In [15], semi-exact methods of HAM, ADM and He's VIM to obtain elastic stress-strain distributions of rotating disks having variable geometry and material properties with thermo-elastic loading is reported. In [16], authors investigated the effect of orthotropy and gradient on disk performance and presented solution to the elastic problem of rotating annular FG polar orthotropic circular disks with free and clamped boundary condition. Explicit expressions of the elastic field were reported for power-law gradation. Without assuming yield criterion and associated flow rule, the effect of thickness of compressible and incompressible materials on the fully plastic angular speed of rotating disks is reported in [17] based on finite deformation using Seth's transition theory.

The performance of a rotating disk is proportional to its limit speed and research attempting to increase the limit elastic speed have been widely reported in recent times. Based on Unified Yield Criterion, [18] reported the limit angular velocity and stresses in disks with variable thickness and compared the limit solution for different yield criterion. In [19], based on von-Mises yield criterion and its flow rule, the analytical solution is derived to calculate the elastic and plastic limit speeds for disks with power function thickness variation and having rigid inclusion. In [20], the authors investigated the performance of hyperbolic annular disk based on von-Mises yield criterion and its associated flow rule, derived the limit elastic speed and reported the influence of disk profile and speed on the size of elastic-plastic zone. A numerical solution based on Galerkin's error minimization principle to investigate the stresses in rotating disks having additional loading in the form of attached masses has been reported in [21]. Authors, in [22], studied the effect of geometry and material properties on the limit speeds of rotating disks with elastic-perfectly plastic and Ramberg-Osgood material model and reported the role of hardening exponent of Ramberg-Osgood equation in determining disk expansion. Limit speed of the disks is reported under different geometry and loading parameters of the disk. Using Tresca's yield criterion, [23] presented exact solution for elastic perfectly plastic FG disks power law material variation and reported limit speed, stresses and displacement. A significant observation of the yield front propagation with increase in speed is also reported. Based on von-Mises yield criterion, [24] reported the elastic–plastic behavior of FG disks for a non-work hardening material and power-law variation of material properties and reported the limit speeds and growth of plastic region at different rotational speeds. The influence of thermo-mechanical loading on stresses and deformation of rotating disk with varying thicknesses under different temperature distribution profiles and the effect of temperature on yield stress and limit speed of the disks has been reported in [25].

The present work reports the limit speeds and stress and displacement states at limit elastic speeds of functionally graded rotating disk. The FG disk is assumed to be treated using powder metal processing by mixing two different metals. For this reason, elasticity modulus, density and yield strength of FG disk vary radially according to the power law function. The problem is modelled by using variational principle, taking the radial displacement field as the unknown dependent variable. Assuming, in such rotating disks, series approximation following Galerkin's principle, the solution of the governing equation is obtained. The validation of the present numerical scheme is carried out with existing literature. Corresponding to various rotational speeds, the stress and displacement field in the disk is estimated. The relevant results are reported in graphical form results obtained from numerical solutions are validated with benchmark results and are found to be in good agreement. The application of variational principle, being an integral formulation treated to result into algebraic expression, yields advantages over classical approaches in terms of simplicity and ease of handling complexities.

2. MATHEMATICAL FORMULATION

An annular circular disk of constant thickness ($\boldsymbol{h_0}$), inner radius ($\boldsymbol{a}$) and outer radius ($\boldsymbol{b}$) is shown in Fig. 1. The disk rotates with an angular speed $\boldsymbol{\omega}$. Rotation induces centrifugal loading resulting into radial displacement governed by boundary conditions of the disk. The disk is symmetric about the axis of rotation and is in a state of plane stress ($\boldsymbol{\sigma_z = 0}$). Due to centrifugal loading, the strain energy (Eq.1) stored in the disk and the external work potential (Eq. 2) corresponding to angular speed $\boldsymbol{\omega}$ is derived.

$$U = \frac{1}{2}\int_V (\sigma\varepsilon)\,dv = \frac{1}{2}\int_V \left(\sigma_\theta\varepsilon_\theta + \sigma_r\varepsilon_r\right)dv \qquad (1)$$

$$V = -\int u\omega^2 r\,dm \qquad (2)$$

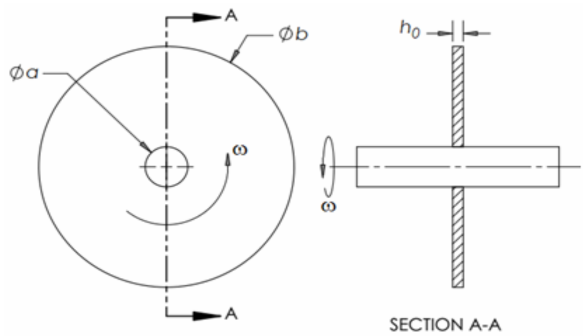

Figure 1. Rotating disk mounted on a rigid shaft: front and sectional side view

In Eq. 2, u is the unknown displacement field. Functionally graded material is produced by pre-determined continuous variation of the constituent materials following a property distribution law defining the variation in volume fractions. Among the popular laws of property distribution, the power-law function and exponential function are the widely used ones to describe the property variation. In the present work, annular disks with power-law function is considered following Eq. 3-4, where $E(r)$ and $\rho(r)$ are the elasticity modulus and density, respectively, at radius r and n is the power index. The formulation could also be extended to other implied distribution functions. The variation of density and modulus of elasticity along the radius of the disk, based on Eq. 3-4, is plotted in Fig. 2.

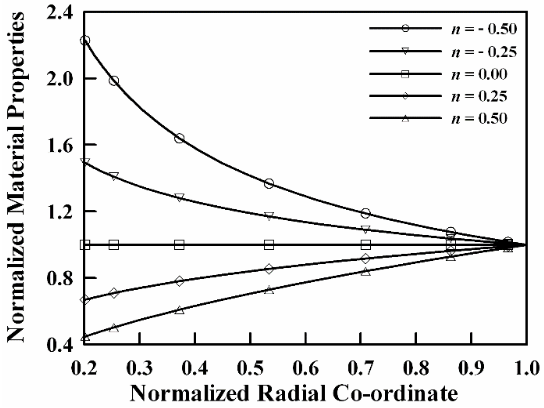

Figure 2. Distribution of material properties at different power indices, *n*.

As it is an established fact that the effect of Poisson's ratio on the deformation state of the disk is insignificant as compared to that of Young' modulus, Poisson's ratio, μ, is assumed to be constant.

$$E(r)=E_0\left(\frac{r}{b}\right)^n \tag{3}$$

$$\rho(r)=\rho_0\left(\frac{r}{b}\right)^n \tag{4}$$

The thin annular disk with in plane (rotational) loading is assumed to be under plane stress condition following constitutive relations given in Eq. 5 and due to rotational symmetry of the geometry, loading and material of the disk, the strain-displacement relations, given in Eq. 6, are used.

$$\varepsilon_r=\frac{\left(\sigma_r-\mu\sigma_\theta\right)}{E(r)};\ \varepsilon_\theta=\frac{\left(\sigma_\theta-\mu\sigma_r\right)}{E(r)} \tag{5}$$

$$\varepsilon_r=\frac{du}{dr}\ ;\ \varepsilon_\theta=\frac{u}{r} \tag{6}$$

Upon substituting the constitutive and compatibility relations in Eq. 1, strain energy becomes

$$U=\frac{\pi}{1-\mu^2}\int_a^b\left\{\begin{matrix}E(r)\dfrac{u^2}{r}+E(r)2\mu u\dfrac{du}{dr}\\ +E(r)r\left(\dfrac{du}{dr}\right)^2\end{matrix}\right\}hdr \tag{7}$$

Based on Minimum Potential Energy Principle (Eq. 8), the solution for the displacement field is obtained.

$$\partial(U+V)=0 \tag{8}$$

Substituting Eq. 2 and Eq. 7 in Eq. 8, the governing equation takes the following form,

$$\delta\left[\begin{matrix}\dfrac{\pi}{1-\mu^2}\displaystyle\int_a^b\left\{\begin{matrix}E(r)\dfrac{u^2}{r}+E(r)2\mu u\dfrac{du}{dr}\\ +E(r)r\left(\dfrac{du}{dr}\right)^2\end{matrix}\right\}hdr\\ -2\pi\omega^2\displaystyle\int_a^b\rho(r)r^2uhdr\end{matrix}\right]=0 \tag{9}$$

To facilitate the numerical computation in non-dimensional space, ξ (= r/b), normalization of Eq. 9 is carried out and the governing equation in normalized coordinate is expressed as:

$$\delta\left[\begin{matrix}\dfrac{\pi}{1-\mu^2}\displaystyle\int_a^b\left\{\begin{matrix}E(r)\dfrac{u^2}{\xi}+E(r)2\mu u\dfrac{du}{d\xi}\\ +E(r)\xi\left(\dfrac{du}{d\xi}\right)^2\end{matrix}\right\}hd\xi\\ -2\pi\omega^2b^3\displaystyle\int_a^b\rho(r)\xi^2uhd\xi\end{matrix}\right]=0 \tag{10}$$

The displacement functions u in Eq. (10), is approximated by a linear combination of sets of orthogonal coordinate functions as

$$u(\xi)=\sum c_i\varphi_i,\ i=1,2,3,.........,n_f \tag{11}$$

The set of orthogonal functions, φ_i, are developed through Gram–Schmidt scheme, in which a start function is used to generate the higher order orthogonal functions. The selected start function must satisfy the essential boundary condition, $u|_a$ = 0 and the natural conditions of radial stress, $\sigma_r|_a$ = 0 and $\sigma_r|_b$ = 0. The start function satisfying the natural and essential boundary conditions is as follows;

$$\varphi_o(r)=\frac{\omega^2r(3+\mu)}{8}\left[\frac{\rho(r)b^2(1-\mu)}{E(r)}-\left\{\frac{\rho(r)(1-\mu^2)r^2}{E(r)(3+\mu)}\right\}\right] \tag{12}$$

Upon substituting Eq. 11 in Eq. 10, the governing equation is obtained in algebraic form:

$$\delta\left[\begin{matrix}\dfrac{\pi}{1-\mu^2}\displaystyle\int_0^1\left\{E(r)\left(\begin{matrix}\dfrac{\left(\sum c_i\varphi\right)^2}{\xi}\\ +2\mu\left[\sum c_i\varphi\dfrac{d\left(\sum c_i\varphi_i\right)}{d\xi}\right]\\ +\xi\left(\dfrac{d\left(\sum c_i\varphi_i\right)}{d\xi}\right)^2\end{matrix}\right)\right\}hd\xi\\ -2\pi\omega^2b^3\displaystyle\int_0^1\rho(r)\left\{\xi^2\sum c_i\varphi\right\}hd\xi\end{matrix}\right]=0 \tag{13}$$

In Eq. 13, the operator, δ is replaced by $\frac{\delta}{\delta c_j}$, $j = 1,2,3,4....n_f$. Using Galerkin's error minimization principle, the following set of algebraic equations is obtained;

$$\frac{1}{1-\mu^2}\sum_{i=1}^{n} c_i \int_0^1 E(r)\left\{\frac{\varphi_i\varphi_j}{\xi}+\left(\mu\left(\begin{matrix}\varphi_i'\varphi_j\\+\varphi_i\varphi_j'\end{matrix}\right)+\left(\xi\varphi_i'\varphi_j'\right)\right)\right\}hd\xi \tag{14}$$
$$=\omega^2 b^3\sum_0^1\int_0^1 \rho(r)\left\{\xi^2\varphi_i\right\}hd\xi$$

Equation 14 can be expressed in matrix form and the solution of unknown coefficients is obtained numerically from $\{c\} = [K]^{-1}\{R\}$ using standard IMSL subroutines and in-house FORTRAN code. The present study is based on von-Mises yield criteria. The solution of Eq. 14, at each load step yields the displacement field and the stresses induced in the disk. The effective stress is, then, compared with the yield stress of the disk at each quadrature point. The angular speed is further augmented till the effective stress induced in the disk reaches the yield value at any location in the disk. The location is termed as yield location, ξ_y and the corresponding speed is called limit elastic speed of the rotating disk. The limit elastic angular speed of the rotating disk, determined with reference to effective stress, evaluated from the principal stresses in the disk using von-Mises failure theory is compared with the yield strength of the material.

Limit state analysis of functionally graded structures is based on the assessment of yield stress at quadrature points along the disk. As the elasticity modulus is a continuous function of radial coordinate, the strain energy at each quadrature point derived from the area under the stress-strain curve also depends on grading index, n. The yield point in the stress-strain curve at each quadrature point along the radial coordinate may experience a shift as shown in Fig 3. The magnitude of the shift in yield point is ascertained from the parameter defining the effective mechanical property of functionally graded structures. This parameter, defined as normalized stress-strain transfer parameter (q), determines the effective modulus and depends on spatial distribution of the constituents of the functionally graded structure that affects the response and is valid for two different rule-of-mixtures models: the Voigt and Reuss models. If the applied load is assumed to cause equal strains in the different phases in FGM, Voigt model is considered. The total stress is the sum of stresses carried by each phase. On the other hand, when each phase of the FGM carries an equal stress and the total strain is the sum of the net strain carried by each FGM phase, then Reuss model is implemented. In most FGM and composites, the effective modulus exists between these two extreme models and is determined using normalized stress-strain transfer parameter. This parameter is an empirical parameter and depends on various factors at the material composition and structure level. However, the particular nature of its dependence have not yet been ascertained. Although other more complex formulations for determining the effective material properties also exist, but the presently discussed one is flexible and only needs a single parameter to be determined.

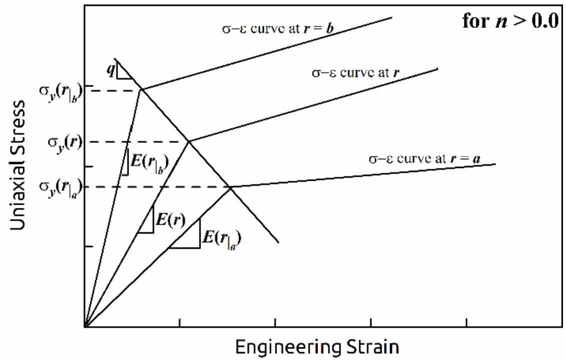

Figure 3. Estimation of yield stress from elasticity modulus along the radius for given grading index, *n*.

The magnitude of this parameter could be determined from experimental data of tensile tests conducted on the FG specimen. For limit state as well as post-elastic analysis, the overall yield stress of the FGM can also be ascertained using the same parameter, [26]. The rule of mixture models could be broadly modeled using distribution functions such as exponential or power law functions for functional grading and can be handled with ease in case of the integral formulations of functional derived under non-linear material behavior. In the present case, as is the variation of elasticity modulus and density, the variation of yield stress along the radial direction is also assumed to follow power law (Eq. 15) with grading index, m.

$$\sigma_y(r)=\sigma_{y0}\left(\frac{r}{b}\right)^m \tag{15}$$

Based on the normalized stress-strain transfer parameter, the relationship amongst the grading indices n and m could be established for a given rule of mixture. However, in terms of distribution functions defining material grading, as in the present study, an arbitrary relation could well be considered. The present study is carried out for $n = 4m$ [24]. The effective stress, at each radial coordinate, is calculated as follows.

$$\sigma_e^2=\sigma_r{}^2-\sigma_r\sigma_\theta+\sigma_\theta{}^2\le[\sigma_y(r)]^2 \tag{16}$$

3. RESULTS AND DISCUSSION

The displacement, strains and stresses of functionally graded annular disk of dimensionless inner radius 0.2 is obtained using Galerkin's error minimization principle as detailed in the preceding section. The numerical values of E_0, ρ_0 and σ_{y0} for the disk material at the outer surface is taken as 207 GPa, 7.850 g/ cm3 and 235 MPa respectively. The results are reported at Poisson's ratio, μ = 0.3 for power index, also known as the grading parameter, n ranging -3.0 to 3.0. Following normalized variables are used:

$$\bar{r}=\frac{r}{b},\quad \bar{\rho}=\frac{\rho(r)}{\rho_0},\quad \bar{E}=\frac{E(r)}{E_0},\quad \bar{\sigma}_y=\frac{\sigma_y(r)}{\sigma_{y0}}$$

$$\Omega=\omega b\sqrt{\frac{\rho_0}{\sigma_{y0}}},\ \bar{u}=\frac{u E_0}{b\,\sigma_{y0}},\quad \bar{\varepsilon}=\frac{\varepsilon E_0}{\sigma_{y0}},\quad \bar{\sigma}=\frac{\sigma}{\sigma_{y0}}$$

The proposed methodology is validated with [24] and a very good agreement is reported. The validation of limit elastic speed is reported in Table 1 for the range of grading parameter considered for the study. The results report critical grading parameter for disks of different aspect ratio based on von-Mises criterion.

Table 1: Normalized limit elastic speed (Ω_l) at different grading parameter, *n* and its validation.

n	-0.50	-0.25	0.00	0.25	0.50
[23]	0.9290	1.0062	1.0917	1.1871	1.2946
[Present]	0.9300	1.0067	1.0916	1.1863	1.2926
% error	0.107	0.049	0.009	0.067	0.154

In [23], the existence of critical grading parameter yielding maximum limit elastic speed has been reported on the basis of Tresca's failure criterion. The effect of grading parameter on limit elastic speed, thus, proposes an interesting insight into the structural performance of the rotating disk. In the present work, based on von-Mises criterion, the variation of limit elastic speed with grading parameter at different aspect ratio (***a***/***b***) has been investigated and reported in Fig. 3. Although non-linear, von-Mises criterion is computationally efficient and unlike Tresca's criterion, a single formulation accounts for different possibilities of yielding. Investigation reveals that, for a given aspect ratio, limit elastic speed increases with increase in grading parameter till the critical value, $\boldsymbol{\Omega}_{lcr}$ is attained and the corresponding grading parameter is called critical grading parameter, $\boldsymbol{n}_{cr}$. As up to this point, it is observed that yielding always initiates at inner radius, ***a***. However, beyond the critical grading parameter, yielding of the disk is observed to initiate at the outer radius, ***b***, thus resulting into an inverse relationship between limit elastic speeds and grading parameter beyond the critical value. A better insight on the effect of grading parameter on limit elastic speed of FG disks of different aspect ratio could be gained from the surface and contour plot presented in Fig. 4. The contour plot serves as an important design data to establish the critical grading parameter and the corresponding maximum limit elastic speed for each aspect ratio. It also provides the limit elastic speed for any combination of grading parameter and aspect ratio. Due to the variation of limit elastic speed with grading parameter, as shown in Fig. 3 (a-b), investigation of the stress and displacement state with in the disk becomes relevant. A comparison of stress and displacement with in the disk of aspect ratio, ***a***/***b*** = 0.20 is studied at a base angular speed over a range of ***n*** values given in Table 1 and reported in Fig. 5 (a-c) and Fig. 6 (a-c). The limit elastic speed at ***n*** = -0.5 ($\boldsymbol{\Omega}_l$ = 0.9300), being smallest, is considered as the base angular speed and the stresses and displacements of disks with different grading parameters are compared at this speed.

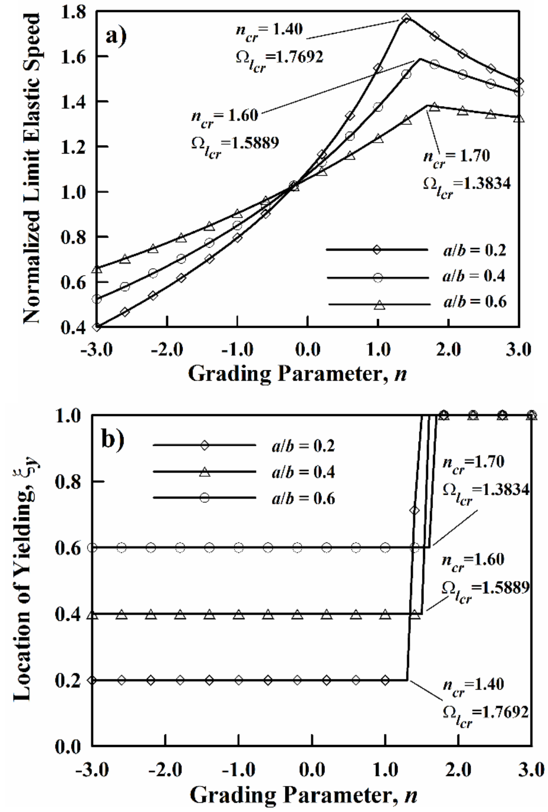

Figure 3. Effect of grading parameter on a) limit elastic speed and b) location of yielding in FG disks

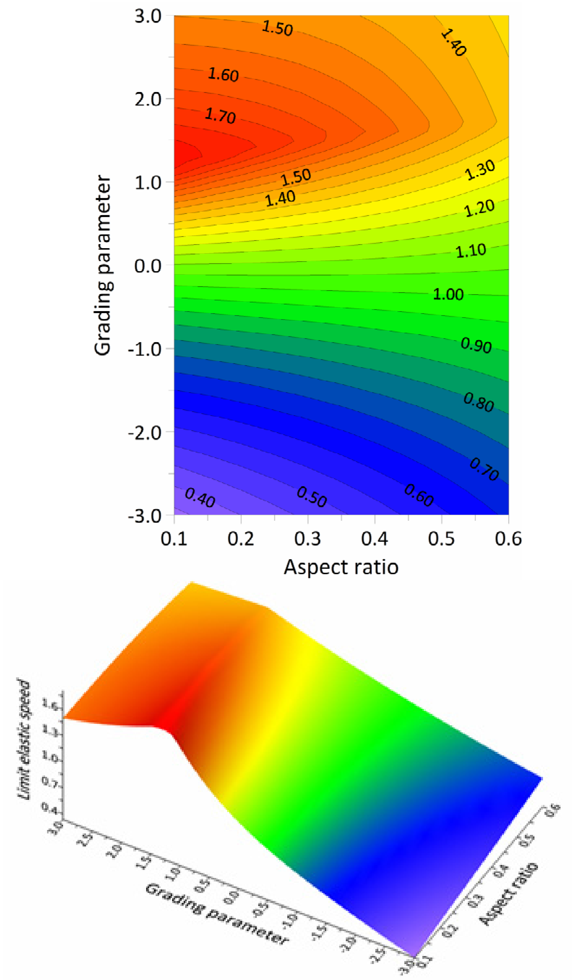

Figure 4. Surface and Contour plot of limit elastic speed with grading parameter and aspect ratio of the disk

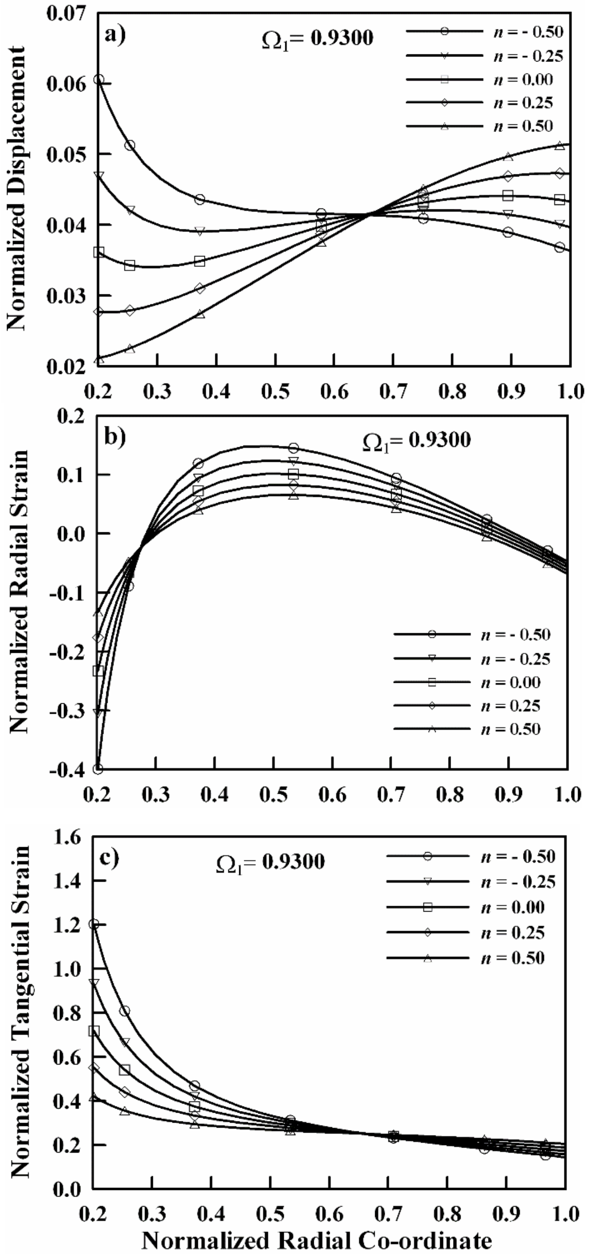

Figure 5. Plots of (a) displacement, (b) radial and (c) tangential strain of FG disk at Ω_l = 0.9300

The distribution of stresses and displacement in disks (***a*/*b*** = 0.20) corresponding to grading parameter and respective limit elastic speed, given in Table 1, is reported in Fig.7 (a-e) and Fig. 8 (a-e). In Fig. 7 (a-e), displacement and distribution of normalized radial and tangential strain is plotted for disks rotating at limit elastic speed for the corresponding grading parameters provided in Table. 1. The distribution of normalized von-Mises, radial and tangential stress for the same is plotted in Fig. 8 (a-e).

4. CONCLUSION

Limit elastic angular speed of functionally graded rotating disk is reported. The study is based on von-Mises failure criterion and solved using series approximation following Galerkin's principle. The functional grading is realized utilizing power law with grading index ***n*** in the range of -3.0 to 3.0.

The results are validated with benchmarks for appropriate system parameter values. Effect of grading parameter on limit elastic speed of FG disks having different aspect ratio is investigated and observed that at each aspect ratio, depending on the location of initiation of yielding, the limit elastic speed increases, reaches a maximum and decreases with increase in grading parameter thus confirming the existence of critical grading parameter at each aspect ratio. The location of yield initiation is also reported in each case as it is observed to play a significant role in optimizing limit elastic speed. Corresponding to limit elastic speed of FG disks at a given grading parameter, the displacement, strain and stress states is also reported. The variational nature of the solution renders the methodology to be easy in handling of the non-linear failure criterion and is observed to be stable, simple and robust. The results are plotted in graphs and could be referred to as important design data.

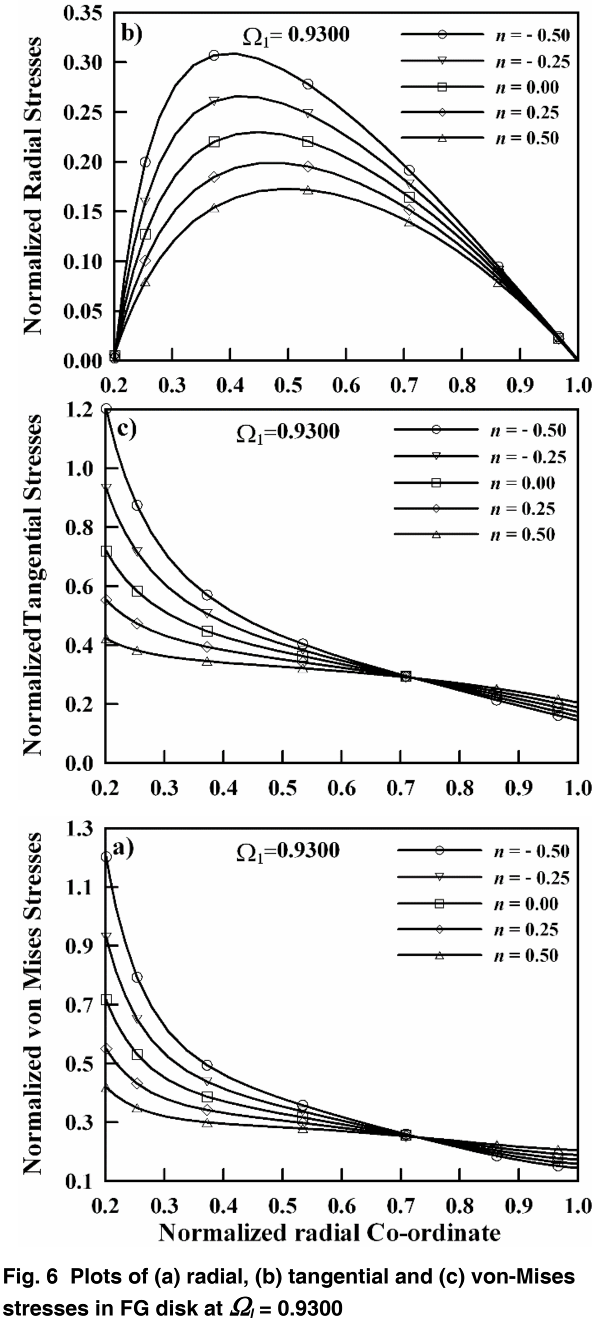

Fig. 6 Plots of (a) radial, (b) tangential and (c) von-Mises stresses in FG disk at Ω_l = 0.9300

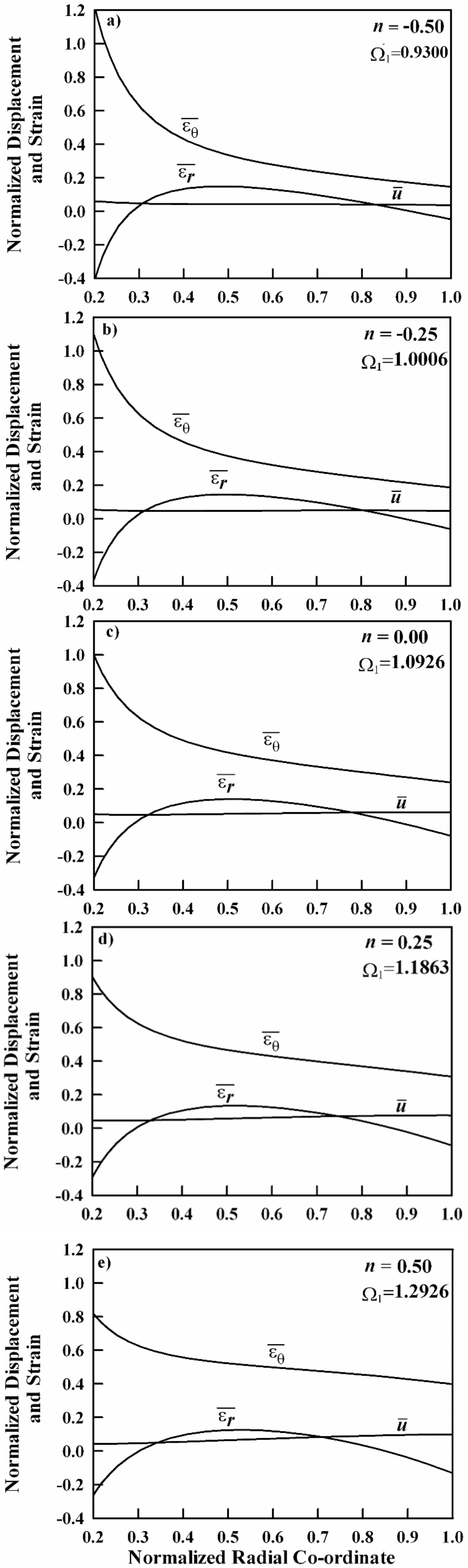

Figure 7. Plot of normalized displacement, radial and tangential strain in FG disks having different grading parameter at respective limit elastic speed

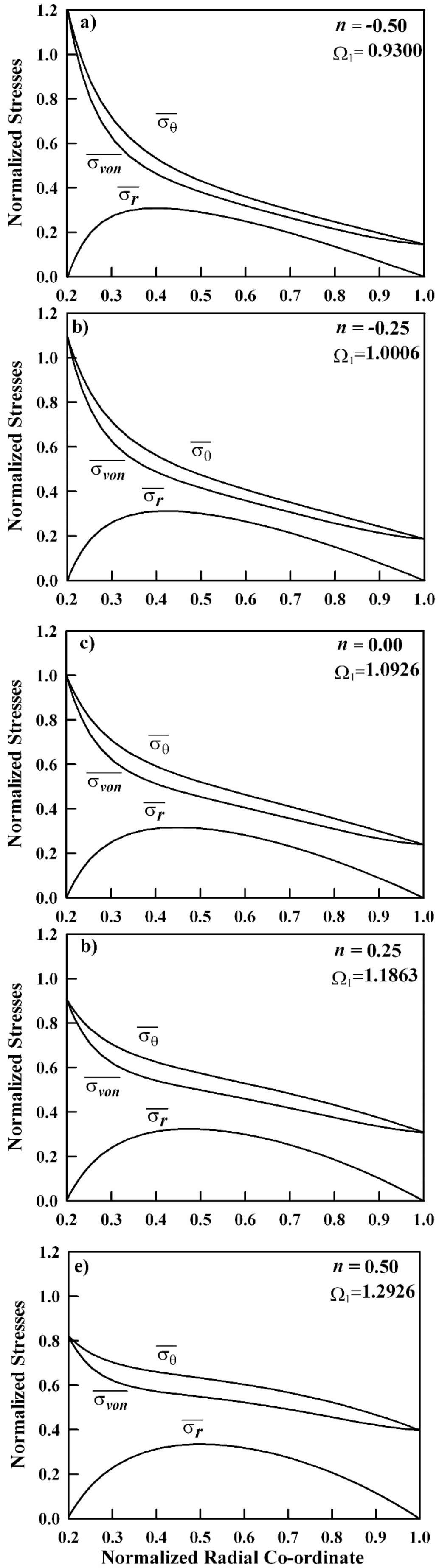

Figure 8. Plot of normalized von-Mises, radial and tangential stresses in FG disks having different grading parameter at respective limit elastic speed

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NOMENCLATURE

a, b, r	Inner, outer radii and radius.
c_i	Vector of unknown coefficients
h	Disk thickness
n	Grading parameter or grading index
n_{cr}	Critical grading parameter
u	Displacement field of the disk
$E_0, E(r)$	Elasticity Modulus at b and r respectively
U	Strain energy of the disk
V	Work potential of the disk

Greek symbols

$\varepsilon_r, \varepsilon_\theta$	Radial and tangential strain respectively
δ	The variational operator
$\rho_0, \rho(r)$	Density at b and r respectively

μ	Poisson's ratio
σ_r, σ_θ	Radial and tangential strain respectively
$\sigma_{y0}, \sigma_y(r)$	Yield stress at *b* and *r* respectively
φ_i	Set of orthogonal polynomials
ξ	Normalized radial co-ordinate
ξ_y	Location of yielding in the disk
ω	Angular speed of the disk
Ω	Normalized angular speed of the disk
Ω_l	Normalized limit elastic speed of the disk

Subscripts

r	radial
θ	tangential
l	at limit elastic state
y	yield related
lcr	critical value at limit elastic state

ГРАНИЦА ЕЛАСТИЧНОСТИ БРЗИНЕ ФУНКЦИОНАЛНО ГРАДИРАНИХ ПРСТЕНАСТИХ ДИСКОВА

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Коришћењем варијационог принципа у радује приказана граница еластичности угаоне брзине ротирајућег диска израђеног од функционално градираног материјала. Полазећи од низа апрокси–мација које следе из примене Галеркиновог прин–ципа, добија се решење основне једначине засноване на фон Мизеовом критеријуму попуштања мате–ријала. Претпоставка је да модул еластичности, гус–тина и напон течења варирају по закону снаге са индексом градирања од – 0,3 до 3,0. При индексу градирања n = 0,0 диск поприма понашање изотропног материјала. Утврђено је да граница еластичности брзине варира у зависности од параметра градирања различитих односа код прстенастог диска и да постоји оптимални индекс градирања за сваки однос. Локација почетка течења постоји у сваком појединачном случају и она игра кључну улогу у оптимизацији границе еластичности брзине. Померај, стање напона и дефор–мације код диска кодграницe еластичности брзина су такође утврђени. Процена резултата је извршена према критеријумима за одговарајуће вредности параметара. На основу варијационог карактера решења и лакоће примене критеријума нелинеарног попуштања утврђено је да је методологија решавања стабилна, једноставна и робустна.