

How Platonic and Archimedean Solids Define Natural Equilibria of Forces for Tensegrity

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The Platonic and Archimedean solids are a well-known vehicle to describe certain phenomena of our surrounding world. It can be stated that they define natural equilibria of forces, which can be clarified particularly through the packing of spheres. [1][2] To solve the problem of the densest packing, both geometrical and mechanical approach can be exploited. The mechanical approach works on the principle of minimal potential energy whereas the geometrical approach searches for the minimal distances of centers of mass. The vertices of the solids are given by the centers of the spheres.

If we expand this idea by a contrary force, which pushes outwards, we obtain the principle of tensegrity. We can show that we can build up regular and half-regular polyhedra by the interaction of physical forces. Every platonic and Archimedean solid can be converted into a tensegrity structure. Following this, a vast variety of shapes defined by multiple solids can also be obtained.

Keywords: Platonic Solids, Archimedean Solids, Tensegrity, Force Density Method, Packing of Spheres, Modularization

1. PLATONIC AND ARCHIMEDEAN SOLIDS

Platonic and Archimedean solids have systematically been described in the antiquity. They denominate all convex polyhedra with regular faces and uniform vertices except the prisms and antiprisms. They obviously have edges of only one length. Platonic and Archimedean solids are considered optimal structures when it comes to the question: How to enclose a maximum volume with an especially low number of uniform edges. It can roughly be stated that the more vertices a polytope has, the more it approximates a sphere. This characteristic is known as sphericity [3]. If we follow a mechanical approach we can suppose, that Platonic and Archimedean solids are configurations based on a minimal potential of energy. Nature is known for finding optimal forms during a long term of development. Since we find some of the previously described forms in environments with rather uniform and constant forces, like in the world of Radiolaria [4], we can consider this as a validation of the assumption [5].

2. PACKING SPHERES

The first mathematician who published an essay on the subject how equal spheres can be packed in such a way that they fill up a given space as much as possible, was Johannes Kepler back in the 17th century [6]. It is a well-studied question, which is relevant for topics like math, crystallography, but also biology and engineering. The densest packing of spheres is related to the so-

called “kissing number” problem. The kissing number problem is asking for the maximum possible number of congruent spheres, which touch another sphere of the same size without overlapping. In three dimensions the kissing number is 12. This number corresponds to the vertices of the icosahedron and the faces of the dodecahedron, relates to the cuboctahedron and the rhombic dodecahedron (a Catalan solid with a single edge length), or as a part, the tetrahedron.

There are several common types of sphere packings. Body-centered cubic structures, face centered cubic structures and hexagonal dense packed structures. There are different scenarios of packing spheres, from which we will only consider densest packings of spheres of the same size [7].

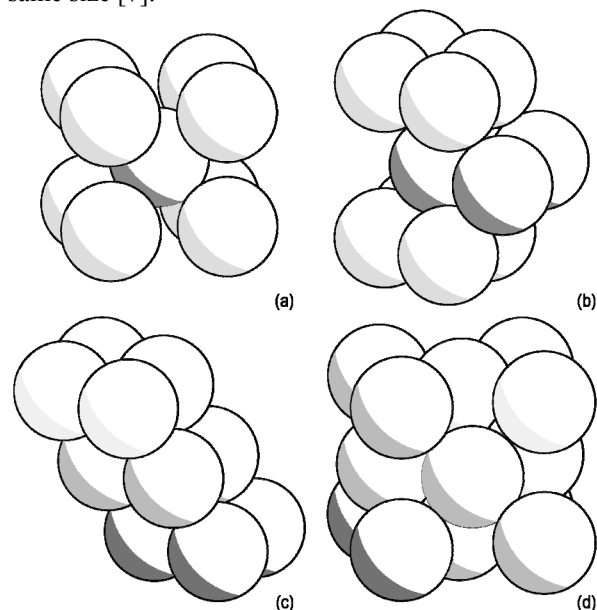


Figure 1. Packing Spheres: BCC (a), HCP (b), FCC (c), FCC (d)

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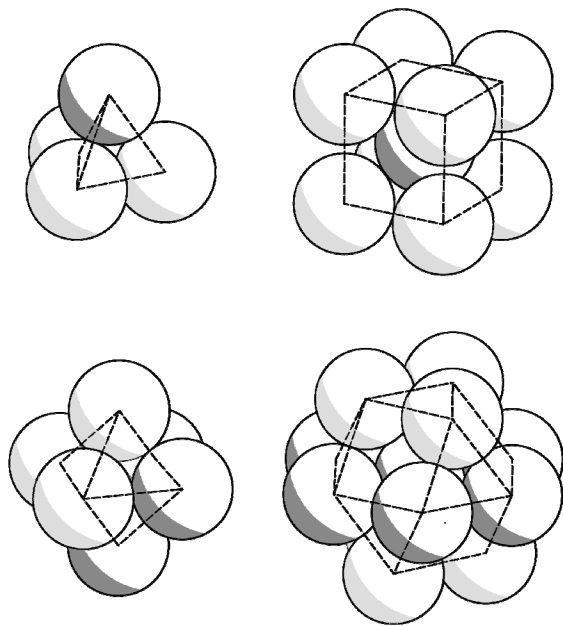


Figure 2. Polyhedra: Tetrahedron, Cube, Octahedron, Cuboctahedron

The hexagonal structures are based on two dimensions on a hexagon where in each corner lays a circle with the radius defined by half of the length of the edges. In the center of the polygon fits a circle of same size such that it touches each of the other six circles. If we form a hexagonal grid, this defines the first layer of dense packed spheres in three dimensions. A second layer is packed onto the first layer such that each sphere of the top layer rolls into a triangular trough of the bottom layer. Not all troughs can be filled which is important for differentiation. Now we have two options to stack the next layer. If we continue to lay the third layer in the same manner as the second we have a so-called ABC layering which is the face centered cubic structure (FCC) shown in Fig. 1(c) and (d). If we lay the third layer such that it is straight above the first, the result is an ABA layering, which is a hexagonal dense packed structure (HCP) as we see in Fig. 1(b). In Figure 1 and 2 the layer A is light grey highlighted, the layer B grey and the layer C in a darker grey. The body-centered cubic structures (BCC), as shown in Fig. 1(a) are formed based on a quadrangular grid where four circles are placed on the corners. In the plane, the circles do not touch. If we set the next layer all troughs are filled. If we continue the layering, eight other spheres touch each sphere. The radius is given by the quarter of the length of the diagonal of the defining cube. To compare the efficiency of the different packings, we use the atomic packing factor (APF) defined by the volume of particles in relation to the volume of the related polyhedron. Of course, that means that the APF can differ for the same structure of spheres. The lattices of spheres which achieve the highest average density is the FCC. Most of the Platonic and Archimedean solids can be found in one of the three structures by connecting the centers of spheres. Tab. 1 shows on which layering the structure is based, the atomic packing factor and, as information, the amount of edges or vertices to compare the structures. Interesting is, that the icosahedron and the dodecahedron have a centric sphere which is

touched by 12 other spheres. For this two polyhedra, there is no configuration with equal spheres, where every adjacent sphere is touching each other. It is nevertheless a structure, which is often found in nature because it defines a near ideal equilibrium of forces. We will show that the icosahedron has a more efficient structure than the cuboctahedron, although the cuboctahedron has the better APF. Based on the potential of forces. Another comparative size shall be the ratio of the edge length to the distance from the vertices to the mass center. The maximum 1 is reached, if we cut a polyhedron out of the HCP or FCC so that the “kissing number” is reached, what is shown by the cuboctahedron in Fig. 2 [7].

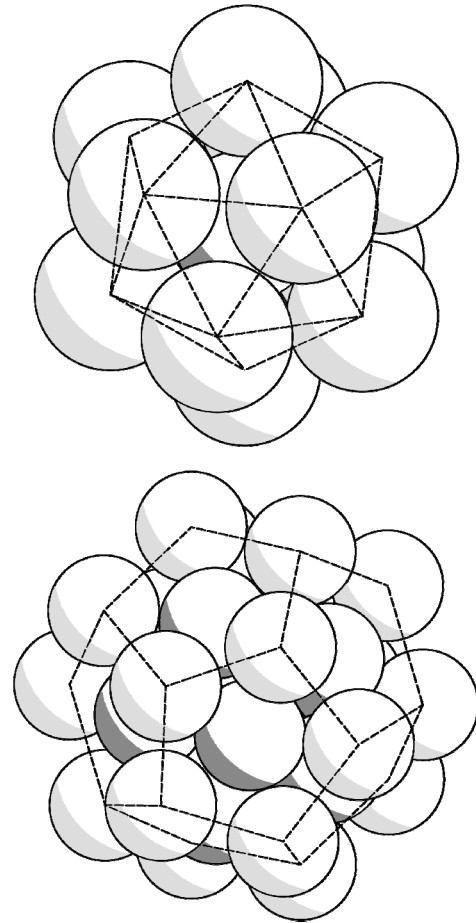


Figure 3. Polyhedra: Icosahedron, Dodecahedron

3. INTRODUCTION TO TENSEGRITY

A tensegrity system has a stable state of equilibrium and works independently from gravity. The system can be preloaded. It consists of ropes or a flowing fabric, on which only tensile forces act. And a second group of compressed components that absorb the pressure forces and moments, but never tensile forces. The compressed components do not touch each other and are connected either at their ends forming a framework or even over the entire element with extensive tension elements. We can divide tensegrity structures into different degrees of "purity". The purest are the structures, where no strut touches another and where only linear members are involved. In the broadest sense, however, any systems in which any configuration of

elements under pressure is kept in balance by a network of ropes, membranes or any tensile elements may be referred to as tensegrity. Pneumatic structures can also be described as a tensegrity structure, e.g. a pumped-up football. This involves the skin as a tension element and the air molecules inside as the pressure elements. Since molecules are modeled as a system of spheres, all the geometric patterns of consecutive spheres become spheres. So, it is possible to represent the configuration of a tensegrity with spherical packages. There is a special group of tensegrities in which all struts have the same length and all cables either. Such structures are called regular tensegrity structures [5,8].

There is always the possibility to reach an equilibrium by constructing tensegrities with a pressure center. Therefore, every corner is connected with the mass center to build a strut and every edge is a cable. Within this configuration struts and cables have the same potential and define an especially even construction so as the presented polyhedra do. Another way is to downscale the used polyhedra and connect the equivalent corners. We have an outer skin defining the polyhedra and an inner tension layer which produces the tensegrity configuration. If the inner layer has a uniform potential, the struts will intersect in the mass center. We need an uneven force distribution, so that the struts do not touch themselves anymore. We need three force vectors to hold a point in three dimensions. We can reduce the inner polyhedra so that every knot is held by two cables and one strut. With the direction of the vectors and their lengths as normalized force the knot is stable, if the sum of the vectors has the length zero. If we look at this with a geometrical approach we see, that an equilateral triangle defines the most uniform vector equilibrium in a plane with three angles of 120° . The edges of a triangle shown as vectors illustrate that the sum is zero. The angle of 120° is described by the three vectors oriented on the lines connecting the center of a triangle with the corners where the center describes the knot to hold. The angle between the three vectors should never be 90° or less. The principle can also be applied in a three-dimensional structure. Based on this knowledge we claim that the polyhedra with triangle faces have the more stable configuration. The less stable configuration is the cube. To define the outer skin, we need four elements because the vectors or edges lay orthogonal. Of special interest are the polyhedra a higher amount of faces and a good ratio of faces to edges like the icosahedron and the dodecahedron. They can be represented by a more efficient equilibrium because two knots in the outer skin can be stabilized by one strut. [1,5]

4. FORCE DENSITY METHOD

To show that there is a stable equilibrium within a Platonic or Archimedean solid, we will use the force density method which is defined by:

$$NC^TQC = F(n) \quad (1)$$

Where N is a matrix that saves the coordinates of the nodes, C is a binary like matrix in which all information about the connectivity of elements is stored and where

diagonal matrix Q defines the force densities. Every force density q_i is one entry in Q . The force density is a quotient of the magnitude of the force or the normal stress and the length of action of the force. Respectively the distance of two neighboring nodes. The equation system established for the solution is linear and allows a fast and numerically stable determination of the equilibrium form. The external forces $F(n)$ are not necessary in our case. The part we want to look at is the Connectivity matrix $D = C^TQC$. It is known that a system is stable, if the rank of Matrix D is $r_D = n - 4$ where n is the number of nodes. In this case, four nodes that parametrize the results are sufficient to create a spatial, reticular, and self-supporting system. In order to check whether a structure developed by the force density method is stable, one has to check the matrices D for their definiteness. If all eigenvalues are greater or equal to 0, the matrix is positive-definite and the structure is stable. Matrix D saves all information about connectivity and about the qualitative distribution of forces. Therefore, for every eigenvalue of D , which becomes zero one rank will be dropped. We can analyze the Connectivity Matrix through the characteristic polynomial. The roots of the characteristic polynomial are the eigenvalues, so we can look under which condition three of them become zero. The force density method was invented to calculate big membrane like structures. In this case, there are held knots and free knots. In (1) they would be separated. Nevertheless, tensegrity have no held knots so we don't need to separate (1). For a more detailed explanation, read article [9] or [10]. A detailed discussion of the derivation would go beyond the scope. We can construct Matrix C directly after following conditions:

$$C_{ij} = \begin{cases} 1, & \text{if element vec. } m_i \text{ ends at knot } n_j \\ -1, & \text{if element vec. } m_i \text{ begins at knot } n_j \\ 0, & \text{if there's no connection} \end{cases} \quad (2)$$

The Rows represent the elements and the columns the knots. In every diagonal element of D there is one force density indicated related to the corresponding element. Normally to use the force density method there are two approaches. Either one defines all force densities and calculates the node location or you find the force densities for specified knot locations. Our method will be explained in chapter 6 [9,11,[12].

5. EQUILIBRIUM MATRIX

The equilibrium matrix stores a lot of important information about a system. It is helpful to analyse this matrix to understand a given system and its configuration. The number of bias states s , the number of movement possibilities m and the pretension states can be determined by analyzing this matrix. It has the dimension (axe) , where e indicates the number of elements of the structure and the number of nodes divided into all coordinate directions. The matrix is filled with the normalized element vectors. Figure 4 shows the construction of the Matrix for a three-dimensional example system.

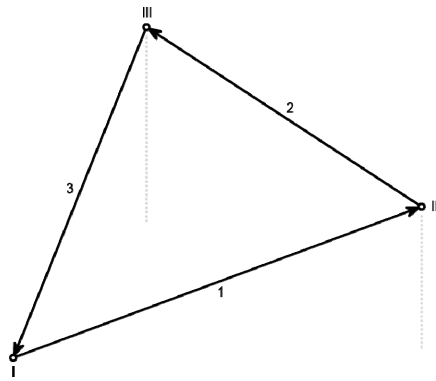


Figure 4. Equilibrium Matrix

$$A = \begin{bmatrix} \frac{(x_I - x_{II})}{l_1} & 0 & \frac{(x_{III} - x_I)}{l_3} \\ \frac{(y_I - y_{II})}{l_1} & 0 & \frac{(y_{III} - y_I)}{l_3} \\ \frac{(z_I - z_{II})}{l_1} & 0 & \frac{(z_{III} - z_I)}{l_3} \\ \frac{(x_I - x_{II})}{l_1} & -\frac{(x_{II} - x_{III})}{l_2} & 0 \\ \frac{(y_I - y_{II})}{l_1} & -\frac{(y_{II} - y_{III})}{l_2} & 0 \\ \frac{(z_I - z_{II})}{l_1} & -\frac{(z_{II} - z_{III})}{l_2} & 0 \\ 0 & \frac{(x_{II} - x_{III})}{l_2} & -\frac{(x_{III} - x_I)}{l_3} \\ 0 & \frac{(y_{II} - y_{III})}{l_2} & -\frac{(y_{III} - y_I)}{l_3} \\ 0 & \frac{(z_{II} - z_{III})}{l_2} & -\frac{(z_{III} - z_I)}{l_3} \end{bmatrix} \quad (3)$$

In Figure 4 no bearing is available. If one of the nodes is a fixed bearing, all three rows of the associated node are deleted from the equilibrium matrix. In order to determine s and m , one has to determine the rank of the node equilibrium matrix. For this purpose, the singular value decomposition of the matrix A is proposed. From the singular values, geometrical properties of the structure can be directly derived. These are useful in assessing the usefulness of tensegrity structures. The singular value decomposition is defined as followed:

$$A = U \Sigma V^T \quad (4)$$

The diagonal matrix $\Sigma = (s_{ij}) \in \mathbb{R}^{m \times n}$ contains the singular values of A . The number of singular values other than zero is equal to the rank r_A of matrix A . Since for many structures there are no singular values that are equal to zero, values that fall below a defined bound are considered zero. The tolerance must be reselected for each structure. Pellegrino (1993) suggests that all values less than the largest singular value times 10^{-3} be considered zero [13]. In further investigations one recognizes that for some structures the tolerance must be selected clearly higher. With the rank now determined, the values s and m are as follows:

$$s = e - r_A \quad (5)$$

$$m = 3n - c - r_A \quad (6)$$

Here, c is the number of degrees of freedom held. Since tensegrity structures have only articulated nodes, the number of supports for these is three times c . If some nodes are stored, you have to adjust the degrees of freedom accordingly. The singular-value decomposition has the following form:

$$A^{m \times n} = \begin{bmatrix} \cdots & u_i \\ & \vdots \\ \cdots & u_m \end{bmatrix} \in \mathbb{R}^{m \times n} \quad (7)$$

$$\text{diag}(\sigma_1; \cdots; \sigma_m) \in \mathbb{R}^{m \times n} \diamond \begin{bmatrix} \cdots & v_j \\ & \vdots \\ \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$

You can now read the rank of the matrix, since the sum of the nonzero values is equal to the rank. Tensegrity structures have normally one pretension condition. In the case that the system is sustainable, the pretension state can be read from the right-singular vectors of A , which is the last column of the matrix V^T . Furthermore, one can read from the decomposition products in which direction a force has to act in order to initiate a movement of the structure. This information is contained in the last column of the matrix U . Because it is an attribute of tensegrity structures, that they can be stabilized by pretension, there is always an infinitesimal possibility of movement. The most important information we got from the calculation is whether the system is stable or not. This is the case when at least the last singular-value becomes zero. The pretension condition must not be confused with the force densities. To come from the pretension to the densities we have to divide them with the lengths of the elements [11,12].

6. METHOD

The investigation of Platonic and Archimedean tensegrities will follow the upcoming steps:

1. Group the elements so that there is an especially low number of different force densities.

Our preferred configuration would be a system with only one coefficient for struts and one for cable. Sometimes you cannot reach the configuration you want with only two different coefficients. In this case we have to find groups such as outer cables and inner cables. If we do not see a rhythm in the system we can use the pretension condition out of the analyzation of Matrix A to see which elements should be grouped.

2. Set up the Connectivity Matrix D with general force densities
3. Search the characteristic polynomial defined by $\chi_D = \det(\lambda E_n - A)$ and determine the eigenvalues λ from A .
4. Choose the eigenvalues that should become zero and set them zero.

Now we can choose force densities, which satisfy the chosen equation. It makes sense to choose the values so

that they barely differ. This ensures a more homogeneous structure. We also have the possibility to continue with general values.

5. Solve (1) with general knots and parameterize the results after four of them.
6. Test whether the desired body can be mapped with the calculated equations

There are several reasons why this approach is beneficial. We not only show, that the tensegrities we analyze are stable but also, that there is an infinite amount of other forms which fulfill the same conditions. Furthermore, we can recognize symmetries in the system, which gives us a better understanding of the constructions. With the stability condition of chapter 5 and the analysis of the eigenvalues, we have two ways to validate our results. For some systems the shown method is too computationally expensive so we will use a geometric approach that is validated with the equilibrium matrix.

To compare these different structures, we will always set the biggest factor 1. With this, we can sum up the pretension condition of all members in a structure to show how big the potential energy is which is needed to depict the structure. Also, it is important to calculate the difference between the highest and lowest power, as well as the average force per element. We will always look at the absolute of the forces so that the high of the force is relevant and not if it is a pressure or tension force.

$$F_v = \sum |v_i|, D_v = \max |v_i| - \min |v_i|, A_v = \sum_i |v_i| \quad (8)$$

7. INVESTIGATION

7.1 Tetrahedron

4 Struts, 6 Cables, Proportion is 3,
 $s = 1, m = 6$

At first, we will analyze the tetrahedron shown in Figure 4. We indicate the knots with roman numerals and the element vectors with arabic ones. The numbering and orientation can be found in the drawing. We allocate cables and struts into a group. We assume that the pressure center will always be the average value of the other nodes. The analysis leads to the result:

$$q_c = -\frac{1}{4}q_s$$

or with the ratio of cable to length of struts

Table 1. Polyhedra

Polyhedra	Vertices	Edges	Faces	Form	APF	Edge/Center	Structure
Tetrahedron	4	6	4	Triangle	0.78	0.61	HCP & FCC
Cube	8	12	6	Square	0.68	0.87	BCC
Cube					0.74		FCC
Octahedron	6	12	8	Triangle	0.72	0.71	FCC
Dodecahedron	20	30	12	Pentagon	0.74	0.71	-
Icosahedron	12	30	20	Triangle	0.73	0.95	-
Cuboctahedron	12	24	14	Triangle & Square	0.78	1.00	FCC
Anti-Cuboctahedron	12	24	14	Triangle & Square	0.78	1.00	HCP

$$v_c = -0.41v_s$$

In this case the both coefficients always have the same relationship so it does not matter how they are chosen. If we parameterize after the center knot the end result is

$$n_5 = \frac{1}{4}(n_1 + n_2 + n_3 + n_4)$$

The formula is satisfied by configurations that have a point symmetry. For every Patonic and Archimedean tensegrity with a pressure center the result will be similar.

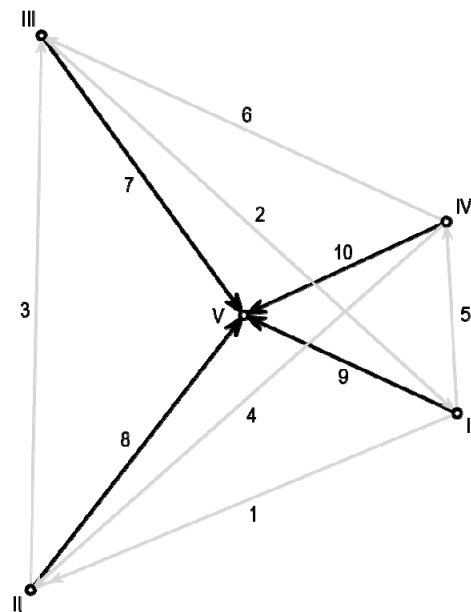


Figure 5. Tensegrity tetrahedron with pressure center

$$C = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

4 Struts, 10 Cables, Proportion is 2/5,
 $s = 1, m = 11$

The next possibility we show is a construction with an inner and an outer tension layer as described in chapter 3. If both layers have the same amount of cables and the same potential, we will have a point symmetric structure where all the struts intersect in the middle. To define the outer edges, we need all cables in this layer. But the inner structure can be reduced. We can remove two cables so that a continuous band is preserved. If we want a structure which fulfills the definition of a classic tensegrity, we have to load the cables in an unrhythmic way. When we review the structure according to chapter 5, we realize how the elements are grouped. Table 2 shows, that due to the calculation there are 6 groups of elements with similar forces. Every value is scaled so that the highest number is set to one. The exact calculation is too computationally intensive so that we limit ourselves to calculate with the equilibrium matrix.

Table 2. Tensegrity tetrahedron with pressure center

	1	2	3	4	5	6
v_i	0.68	0.64	0.49	0.42	0.31	-1.00
q_i	0.67	1	0.15	0.13	0.10	-0.39
l_i	0.31	0.20	1.00	1.00	1.00	0.78

For the tetrahedron we see, that the structure with a pressure center is the better one when we overlook the fact that it is not a pure tensegrity. The benchmark is as follows:

$$F_v = 6.45; D_v = 0.59; A_v = 0.645$$

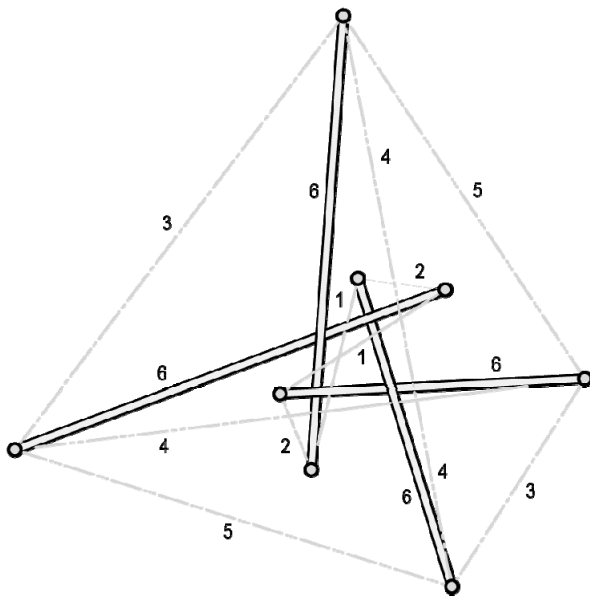


Figure 6. Tensegrity tetrahedron

For the second structure, we see that the total potential is higher without reducing the average potential per element so the first tensegrity structure is technically better.

$$F_v = 9.08; D_v = 0.69; A_v = 0.645$$

7.2 Cube

8 Struts, 12 Cables – Proportion is $2/3$, e_3
 $s = 1$, $m = 8$

In this case, the stability criteria are met under the following conditions:

$$q_c = -\frac{1}{2}q_s \text{ or } v_c = -0.58v_s$$

From now on we will write the solutions as matrices to have a better overview. The four redundant nodes are represented by the rows and the ones to calculate by the columns.

Table 3. Tensegrity Cube with pressure center

	n_5	n_6	n_7	n_8	n_9
n_1	-1		1		-1
n_2		2	2	2	2
n_3	1	-1	-1		
n_4	1		-1	-1	

If we analyze the matrix of Figure 7 we see, that in all cases were three nodes depending on each other they will form a straight line in a cube. If we choose the corresponding vertices to a random cube, the solution will exactly define the boundary. For the cube is it not possible to form a pure tensegrity like in Figure 6. This occurs because the vertices are defined by edges which stay orthogonal to each other. Therefore, the construction can always be deflected. We see, that the total potential of the structure rises compared to the tetrahedral one, which correlates to the rise of elements.
 $F_v = 14.96; D_v = 0.42; A_v = 0.748$

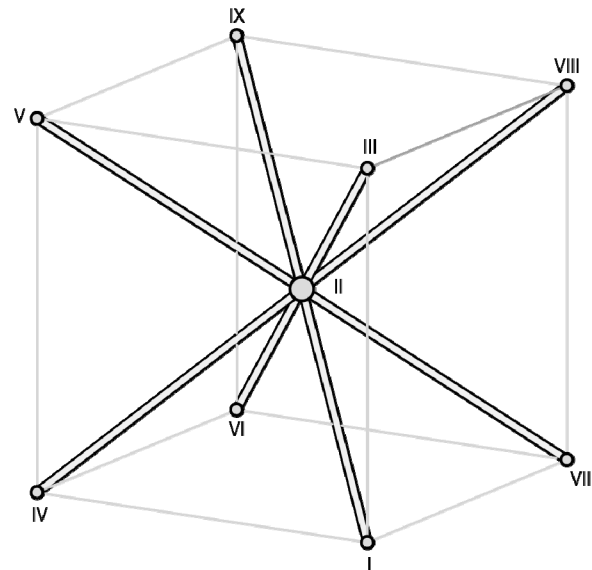


Figure 7. Tensegrity Cube with pressure center

7.3 Octahedron

6 Struts, 12 Cables, Proportion is $1/2$,
 $s = 3$, $m = 6$, knots = 7

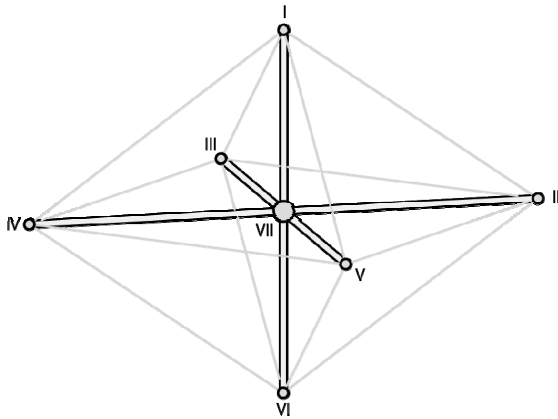
With the tetrahedron, the symmetry properties are not easily recognized. But if we look at it more accurately we also see the symmetry. The centre knot n_7 can be parameterized after three knot pairs. So, it has to lay between every pair of nodes. Also, every defined knot pair has a point symmetry to another pair through the centre point. With this knowledge, we can conclude that

the configuration of the octahedron is contained in the sample space.

$$q_c = -\frac{1}{4}q_s \text{ or } v_c = -0.35v_s$$

$$n_3 + n_5 = n_2 + n_4; n_1 + n_6 = n_2 + n_4$$

$$n_7 = \frac{1}{2}(n_2 + n_4)$$



Fig

Figure 8. Tensegrity octahedron with pressure center

5 Struts, 8 Cables – Proportion is 5/8
 $s = 3, m = 6, \text{knots} = 7$

There is a second possibility, where we can reduce the overall elements (Figure 9). For that we implement a ring of pressure and a center strut. We divide the elements into four groups. The upper layer of cables q_o , the lower layer of cables q_u , the circumferential struts q_s and the central strut q_c .

If the following two conditions are met, the rank of the coefficient matrix is 2 as required for a stable construction:

$$q_s = -\frac{1}{2} \left(\frac{4q_o q_u}{q_o + q_u} \right); q_c = -\frac{4q_o q_u}{q_o + q_u}$$

A possible solution is:

$$q_o = 1; q_u = 1; q_s = -1; q_c = -2$$

This results in the following parameterization:

$$n_5 = n_2 - n_3 + n_4 \quad n_6 = -n_1 + n_2 + n_4$$

With general force densities is the solution:

$$n_4 = \frac{-n_2 q_o + 2n_6 q_o + 2n_1 q_u - n_2 q_u}{q_o + q_u}$$

$$n_5 = \frac{-n_3 q_o + 2n_6 q_o + 2n_1 q_u - n_3 q_u}{q_o + q_u}$$

6 Struts, 16 Cables, Knots 12, Proportion is 3/8,
 $s = 1, m = 13$

The third approach uses an inner tension ring. Therefore, six struts show from the outer layer to the mass centre but are deflected because they should not touch each other to satisfy the classic definition of tensegrity. The groups are coloured and shown in Fig. 13.

Structure with pressure center:

$$F_v = 10.2; D_v = 0.65; A_v = 0.567$$

Structure with main strut:

$$F_v = 5.24; D_v = 0.65; A_v = 0.40$$

$$v_o = 0.35; v_u = 0.35; v_s = -0.35; v_c = -1$$

Pure tensegrity Structure:

$$F_v = 15.74; D_v = 0.833; A_v = 0.656$$

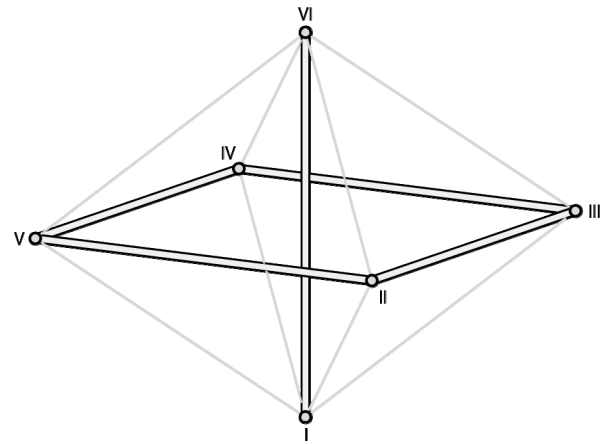


Figure 9. Tensegrity octahedron reduced

In this case, we see how the total potential massively decreases from the first to the second structure. If we include the potential per element we see, that it is not just about the reduction of the elements. The third structure is again less efficient but meets the condition of a pure tensegrity.

7.4 Dodecahedron

20 Struts, 30 Cables – Proportion is 2/3
 $s = 1 - m = 14 - \text{Knots } 20$

The dodecahedron with a centred knot will be tested with the equilibrium matrix to avoid an unclear result and to reduce the computational effort. The analysis of the singular values shows us, that the configuration is absolutely stable. For the cables we get a tensioning factor of 0.9341 and for the struts -1.00 and force densities of -1.00 and 0.7639.

10 Struts, 30 Cables – Proportion is 1/3

$s = 1 - m = 21 - \text{Knots } 20$

The dodecahedron can be represented through a pure tensegrity with only 10 struts and 30 cables. There is exactly one possible option to connect them so that two vertices are connected through one strut. To reach this configuration, no homogeneous distribution of forces is possible. The right singular vector shows us, that we need 6 groups of members, which means that 6 different pretension conditions are needed to achieve the searched configuration.

Table 4. Tensegrity dodecahedron

	1	2	3	4	5	6
v_i	-0.88	-0.50	0.30	0.38	0.61	1.00
q_i	-0.38	-0.22	0.30	0.38	0.61	1.00
l_i	1.00	1.00	0.44	0.44	0.44	0.44

There is another configuration in which all cables have the same force density. But it must be divided into

“horizontal” and “vertical” struts. When the cables have a force density of 1.00 the horizontal struts have a force density of -0.63 and the vertical struts of -0.37. In this case, only the lower and upper polygons are preserved. All the others are deformed.

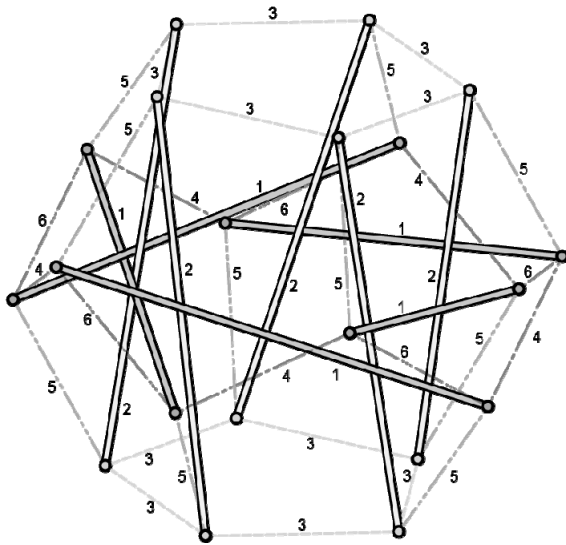


Figure 10. Tensegrity dodecahedron

Structure with pressure center:

$$F_v = 48.03; D_v = 0.066; A_v = 1.20$$

Reduced tensegrity dodecahedron:

$$F_v = 22.87; D_v = 0.697; A_v = 0.572$$

The first structure is very homogeneous, which is shown by the small difference value of the maximal and minimal potential. Nevertheless, the total potential can be reduced substantially in the second structure. Therefore, it is the better and more efficient one.

7.5 Icosahedron

12 Struts, 30 Cables – Proportion is 2/5

$$s = 9 - m = 6 - \text{Knots } 12$$

In this case, the symmetries are clearly visible again. The parameterizing knots define the triangle of one of the 20 faces. The fourth knot is the mass center. We have written the ratios as numerical values to ensure the best possible overview. The first three columns define straight lines and with that the opposite face of the icosahedron. Knot 4, 6 and 8 are one group which are connected with the base face through one cable and thereby define the higher lying vertices. Knot 5, 7 and 9 in their group define the lower vertices connected through two cables. The nodes have the same numbering as in Figure 11, with the central node numbering 13.

6 Struts, 24 Cables – Proportion is 1/4

$$s = 1 - m = 7 - \text{Knots } 12$$

The second more efficient approach is illustrated in Fig. 11. We can not only reduce the number of struts but also the outer cables which define the edges of an icosahedron. At every knot there are 5 elements which have the same orientation of element vectors so the cables and struts can have an even coefficient.

$$q_c = \frac{1}{-(5 + \sqrt{5})} q_s$$

Table 5. Tensegrity icosahedron with pressure center.

	n ₁	n ₂	n ₃	n ₄	n ₅
n ₁₀		-1		1	0,618
n ₁₁			-1	-0,618	0,618
n ₁₂	-1			-0,618	-1
n ₁₃	2	2	2	1.236	0,764

	n ₆	n ₇	n ₈	n ₉
n ₁₀	-0,618	-1	-0,618	0,618
n ₁₁	1	0,618	-0,618	-1
n ₁₂	-0,618	0,618	1	0,618
n ₁₃	1.236	0,764	1.236	0,764

$$q_c = -\frac{2}{3} q_s$$

Table 6. Pure tensegrity icosahedron

	n ₂	n ₃	n ₄	n ₅	n ₆	n ₇	n ₈	n ₉
n ₁	5	-1	-3	4	4	4	4	7
n ₁₀	-4	12	8	1	1	8	8	7
n ₁₁	7	-14	-7	7	-7	-14	-7	-14
n ₁₂	-1	10	9	-5	9	9	2	14

Structure with pressure center:

$$F_v = 16.36; D_v = 0.855; A_v = 0.39$$

$$v_c = 0.1453; v_s = -1$$

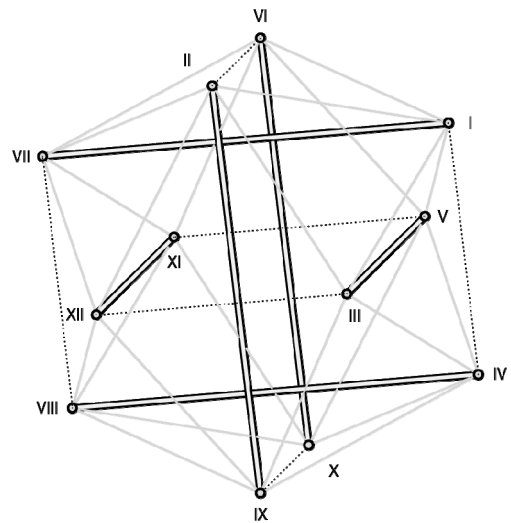


Figure 11. Pure tensegrity icosahedron

Reduced tensegrity icosahedron:

$$F_v = 15.80; D_v = 0.592; A_v = 0.527$$

The difference of the total potential is small in this case. Nevertheless, the distribution of forces becomes more homogenous in the second structure. This makes the structure more effective.

7.6 Cuboctahedron

12 Struts, 24 Cables – Proportion is 1/2

$$s = 4 - m = 7 - \text{Knots } 13$$

As for the examples that have already been shown, there is a stable equilibrium within a construction with a centred pressure knot. The difference is that there is no homogenous partition for cable and strut coefficient. If we want to get the default configuration, we have to divide the elements into 10 groups. The uneven force distribution is caused by the different angles of the element vectors in the outer layer.

12 Struts, 24 Cables – Proportion is ½

$$s = 7 - m = 7 - \text{Knots } 12$$

There is one possibility to build up a tensegrity with an even distribution of cable and strut coefficient. For that we lay 4 isosceles triangles into the cuboctahedron. This creates groups of forces that act in a circle and absorbs the pressure forces.

$$q_c = -\frac{2}{3}q_s$$

Table 7. Homogenous octahedron tensegrity

	n ₂	n ₃	n ₄	n ₅	n ₆	n ₇	n ₈	n ₉
n ₁		-3		3	-3	3	-3	
n ₁₀	2	1	2	-1	2	1	3	-1
n ₁₁	-1	4	2	-1	2	-2	3	1
n ₁₂	2	1	-1	2	2	1		1

The total potential of the first structure is almost half the size of the second structure presented. However, it is virtually unusable, since the differences in the potential of the elements are almost 100 percent.

$$F_v = 13.30; D_v = 0.992; A_v = 0.37$$

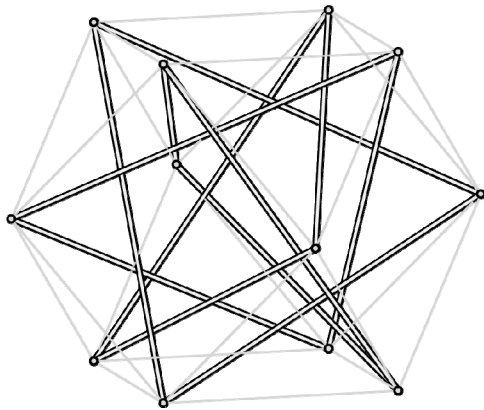


Figure 12-1. Homogenous octahedral tensegrity

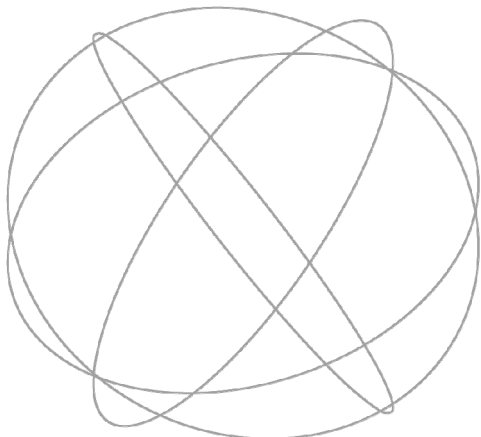


Figure 12-2. Homogenous octahedral tensegrity

Compared with the structures constructed out of the other polyhedra the homogenous octahedral tensegrity has an average difference of potentials so as the potential per element. It is therefore to be classified as a structure that best reflects a cuboctahedron.

$$F_v = 21.24; D_v = 0.615; A_v = 0.59$$

8. BUILDING KITS

Every tensegrity which is identified as stable can be put together and forms a new stable construction. Therefore, the faces which correspond are put together so that the multiple knots lay together. The only thing which is changing is the number of bias states which just get added together. But there is a second method to combine the different construction within which the classic definition of tensegrity is preserved. We put two equivalent faces together and rotate them so that the triangular faces form hexagons or the square gets an octagon. Or in other words, the edges of a surface are split and form new nodes for the overlying element. We connect the polygons through cables and add new cables which draw together the multiple polyhedra tensegrity. Fuller [5] described this proceeding for the tetrahedron as stacking cubic elements as a tower were the vertices of the tetrahedron lay exactly so that they create cubes. The cubic stacking will subside to reach a stable equilibrium.

9. CONCLUSION

In conclusion, it has been proved that each Platonic solid consists of an equilibrium of forces which evolves from a pressure center. Even though the cuboctahedron comprises best sphere packing, it seems to lack in providing most efficient balance of power. This finding could also explain why the shape of the icosahedron is found more often in nature. Furthermore, it has been shown that tensegrities with a higher number of faces are corresponding to bodies which require fewer elements. These pure tensegrities include better potential of forces. Moreover, they show that not necessarily all spheres of a packing have to interact with each other to accept the shape of the corresponding sphere package. Tensegrity structures have established themselves as a particularly efficient structure in some cases. Furthermore, it has been shown that even with smaller order polyhedra tensegrity is constantly presented, besides orthogonal outer edges. Although these are less efficient than those with a pressure center, they can be useful in Bio-Tensegrity and orthopedics to model movements. The idea of packing spheres is useful for finding forms which preserve a low potential of energy. As a consequence, they are particularly stable, and the reduction to polyhedra gives us the possibility to construct modular kits, which can be arbitrarily extended. It has been also demonstrated successfully how the constructed elements can be used as a kit, and these can be made even more efficient by the idea of sphere packing.

Based on the investigations on polyhedra, two options have been identified for analyzing and parameterizing cable-strut-systems and their contained tensegrity. Since you already need information about the structure for this

analysis tools, it seems reasonable to progress from a geometric development of tensegrity to a force-controlled one. These inquiries, especially about the assembly of unit cells, can be continued for an expanded period of time by using these methods. All calculation methods are programmed for Grasshopper, whereby an efficient analysis tool for constructing tensegrity is available [14].

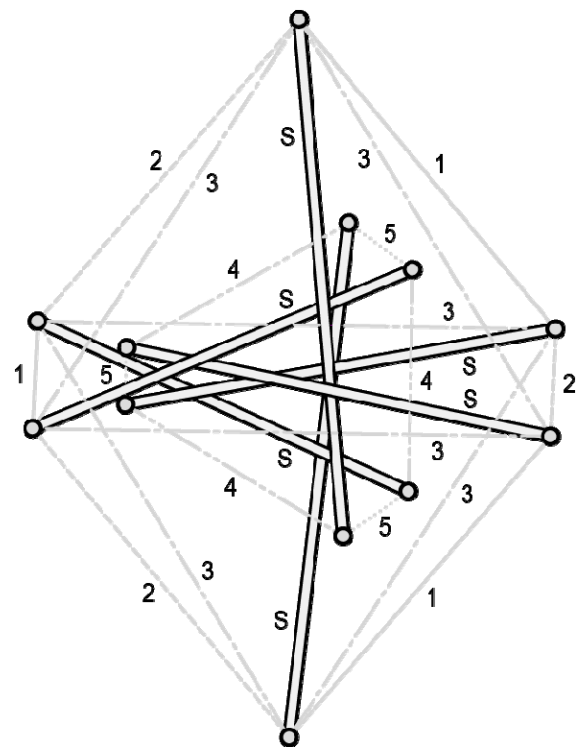
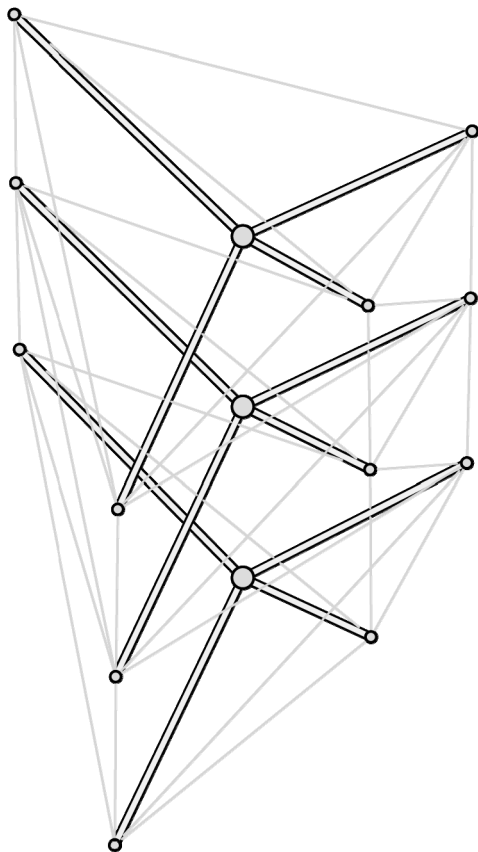
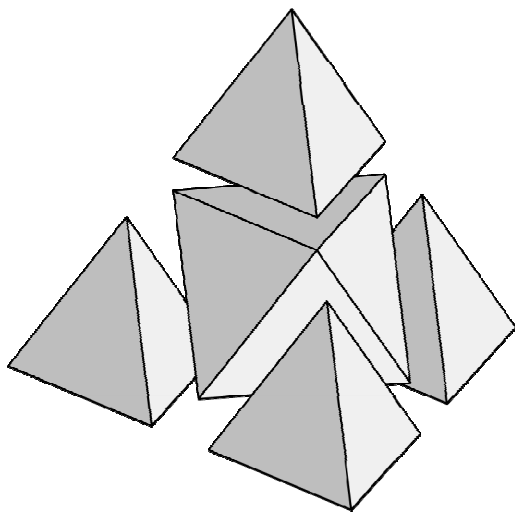
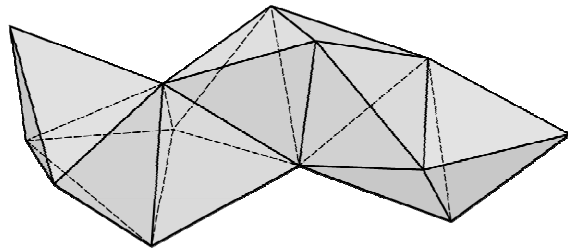


Figure 13. Building Kit & octahedron

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NOMENCLATURE

q_k	force density of element k
n_j	node j
r_D	rank of matrix D
C	connection matrix
Q	force density matrix
N	matrix of nodes
D	connectivity matrix
$F(n)$	external forces at node n
e	number of elements
r_a	rank of matrix a
s	number of bias states
m	movement possibilities
A	equilibrium matrix
U	left-singular vectors

Σ	diagonal matrix of singular values
n	number of knots
c	number of degrees of freedom

КАКО ПЛАТОНОВА И АРХИМЕДОВА ТЕЛА ДЕФИНИШУ ПРИРОДНУ РАВНОТЕЖУ СИЛА ТЕНЗИГРИТЕТА

М.Ф. Ајхенауер, Д. Лордик

Платонова и Архимедова тела добро су позната средства за описивање одређених феномена нашег окружења. Може се рећи да они дефинишу природну равнотежу сила која се може посебно објаснити кроз паковање сфера. [1] [2] Да би се решио проблем најгушћег паковања, може се користити и геометријски и механички приступ. Механички приступ заснива се на принципу минималне потенцијалне енергије, док геометријски приступ користи минималне удаљености тежишта. Врсте крутих тела дефинисане су центрима сфера. Ако ову идеју проширимо реактивном силом, која делују ка споља, добијамо принцип тензег-ритета. Можемо показати да је могуће изградити правилни и полу-правилни полиедар интеракцијом физичких сила. Свако Платоново и Архимедово тело може се претворити у тензигритетску структуру. Након тога, може се остварити и велика разноликост облика дефинисаних мултипликацијом појединачних полиедара.