Finite Element Model of Circularly Curved Timoshenko Beam for In-plane Vibration Analysis

Azin Nadi
School of Mechanical Engineering, Shiraz University, Shiraz, Fars
Iran

Mehdi Raghebi
Department of Mechanical Engineering, University of Birjand, Birjand, Southern Khorasan
Iran

Curved beams are used so much in the arches and railway bridges and equipments for amusement parks. There are few reports about the curved beam with the effects of both the shear deformation and rotary inertias. In this paper, a new finite element model investigates to analyze In-Plane vibration of a curved Timoshenko beam. The Stiffness and mass matrices of the curved beam element was obtained from the force-displacement relations and the kinetic energy equations, respectively. Assembly of the elemental property matrices is simple and without need to transformation matrix because of using the local polar coordinate system. The natural frequencies of curved Euler-Bernoulli beam with large thickness are not sufficiently accurate. In this case, using the curved Timoshenko beam element is necessary. Moreover, the influence of vibration absorber is discussed on the natural frequencies of the curved beam.

Keywords: *Curved Timoshenko Beam, In-plane response, Finite Element Method, Vibration Absorber.*

1. INTRODUCTION

Static or dynamic analysis of curved structures such as arches, rings, shells of rotating machinery, and railway bridges is one of the common engineering issues. One of the essential structural components are beams, which are widely used in both macro-scale like composite laminates [1-4] and micro/nano systems as sensors [5–8] and actuators [9]. One way for modelling these structures is by using a finite element numerical method. However, equations of governing relations on the finite element model of curved beams element have not been used a lot; maybe the complexity of formulas that are available for the curved beam element is one of the reasons for this subject [10]. Therefore, one of the aims of this research is to offer a simpler formulation for analysing the in-plane vibration of the curved beam. Some researchers have studied the curved beam that some of the important works will be mentioned.

Lin and Lee [11] analysed the dynamic response of circular Timoshenko beams with general elastic boundary conditions based on closed-form solutions. The free in-plane vibration analysis of a circular curved beam by a systematic approach was investigated by Kang et al. [12]. The in-plane vibration of curved beam structures was investigated by Chen [13] based on the differential transform method. To predict the in-plane free vibration of a large deflected pre-stressed cantilever curved beam (the Euler-Bernoulli beam) fixed at both ends, Ozturk [14] introduced the reversion and finite element methods with a straight-beam element approach. In this research, the reversion method is used to obtain the non-linear deflection curve of the flexible beam undergoing large deflection. Eisenberger and Efraim [15] presented an exact dynamic stiffness matrix for a circular beam with a uniform cross-section and two different boundary conditions. The matrix derived from the differential equations of motion for the beam and was free of membrane and shear locking as the shape functions. Huang et al. [16] derived the in-plane and the out-of-plane transient response with two different boundary conditions. In this research, the dynamic stiffness matrix method and the numerical Laplace transform were used for the non-circular Timoshenko curved beam. Leung and Zhu [17] investigated the in-plane vibration of thin and thick curved beams with classical boundary conditions based on the finite element method. Several Fourier p-elements for in-plane vibration of thin and thick curved beams having a uniform and non-uniform cross-section presented. In this research, the elements with enriching shape functions avoided membrane and shear locking. In-plane free vibration of circular curved Timoshenko beam based on Chebyshev polynomials was investigated by Lee [18]. In this research, the pseudospectral method and basis function for the boundary conditions was used. To determine the natural frequencies, Kim et al. [19] developed a thin circular beam based on the finite element by considering the effects of shear deformation and rotary inertia. The stiffness and mass matrices are derived from the strain energy and kinetic energy, respectively. The local polar coordinate system was used for developing the matrices and transformed into a global Cartesian coordinate system for assembling. Yang et al. [20] studied the free in-plane vibration of uniform and non-uniform curved beam by considering the effects of axis extensibility, rotary inertia, and shear deformation. In this research, the differential equations are derived by using the extended-Hamilton principle and solved numerically using the Galerkin finite element method.

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Correspondence to: Dr Mehdi Raghebi
Department of Mechanical Engineering, University of Birjand, Birjand, Southern Khorasan, Iran
E-mail: raghebi@birjand.ac.ir
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Wu et al. [21] derived the un-coupled equation of motion for the circumferential displacement of an arch structure to analyse the free in-plane vibration. In this research, they obtained a frequency equation by using the compatible equations for the displacements and the equilibrium equations for the forces and moments at each intermediate node and two ends of the entire curved beam. Wu and Chen [22] presented a technique to replace all complex coefficients of the eigenvalue equation by the real ones for the natural frequencies and mode shapes of out-of-plane free vibrations of a uniform curved Euler-Bernoulli beam in various boundary conditions. Moreover, they compared the results with the approximate ones obtained from the finite-element method. Caliò et al. [23] investigated the natural frequencies and vibration modes of structures obtained by an assemblage of circular Timoshenko beams. In this research considered both the in-plane and out-of-plane motions. Moreover, a parametric analysis of the in-plane and out-of-plane dynamic behaviour of the single arch was performed. Talukdar and Roy [24] modelled a curved Timoshenko beam and analysed in-plane free vibration of the cracked curved beam with both ends fixed conditions. Yang et al. [25] studied different approaches to solving the developed equations of motion of a curved beam. This research considered the effect of the shear deformation, rotary inertia, and the axial extensity and the differential equations of motion for the curved beams discussed. Liu and Zhu [26] developed an efficient formulation of a circular Timoshenko beam for static, vibration, and wave propagation problems based on wavelet-based finite element models of in-plane and out-of-plane mo–tions of circular beams according to Hamilton's prin–ciple. Lv et al. [27] presented a solution for the in-plane vibration of multi-span curved Timoshenko beams with general elastic boundary conditions by combining with the improved Fourier series method and Rayleigh-Ritz technique. Lee and Yan [28] presented a simple method for finding the analytical solution for natural frequ–encies of a curved Timoshenko beam in out-of-plane motion with non-linear boundary conditions based on the shifting function method. In this research, three coupled governing differential equations were derived via Hamilton's principle. Liu et al. [29] addressed In-plane and out-of-plane free vibration analysis of Timo–shenko curved beams based on the isogeometric met–hod, and a practical scheme to avoid numerical locking in both of the two patterns is proposed in this paper.

Davis et al. [30] obtained the stiffness and mass matrices of the Timoshenko curved beam for In-plane vibration with the force-displacement relationships and kinetic energy equations, respectively. In this paper, all matrices that are based on the local Cartesian coordinate system are written for a direct beam. In this case, before the assembling element, they should convert matrices by using transformation matrices from the local Cartesian coordinate to the global coordinate even in the case of the curved beam curvature is constant. Lebeck and Knowlton [31] were obtained stiffness matrix of three-dimensional curved beam element from the force-displacement relationships by ignoring the effect of shear deformation. In their research, the in-plane beam movement is coupled with the out-of-plane movement because of the asymmetry of beam cross-section. Moreover, the stiffness matrix based on the local polar coordinate system was obtained. Therefore, the assembling element it is not necessary to convert coordinates.

Palaninathan et al. [32] obtained the stiffness matrix of the Timoshenko curved beam in three- dimensional state by Castigliano theorem. Moreover, they considered the coupling effects between the vertical and lateral shear forces. The stiffness matrix was written in this paper like [30] based on the local Cartesian coordinate system for a direct beam. Jong-Shyong Wu et al. [33] investigated the In-plane vibration of a curved Classical beam (Euler-Bernoulli beam). In this research, the mass matrix of the curved beam was calculated with con–si–deration of the rotary inertia effect with moving load on the curved beam. Petyt et al. [34] extracted the mass and stiffness matrices of the curved beam element based on the two-displacement function by ignoring rotary inertia effects.

In the present research, by using methods described in [30, 31], stiffness and mass matrices of the curved beam element will be obtained from the force-displa–cement relationships and the kinetic energy equations, respectively. The method has been used in the present study has the following advantages compared to the previous articles:

1-Instead of using the local Cartesian coordinate described in [30], in this paper, the local polar coordinate system will be used. As a result, for a cir–cular curved beam with constant curvature, the stiffness matrix can be obtained by assembling directly without the coordinate transformation matrices.

2-Although the method presents in this paper for obtaining the stiffness matrix somewhat similar to the method presented in [31], this paper considers the shear deformation effects, and the mass matrix will be obtained. It would appear that by composing the met–hods explained in [30, 31], a third formulation will be presented. It is similar to formulation was derived by Wu et al. [35] for analysing the out-of-plane vibrations of curved beams.

2. FORMULATION AND METHODOLOGY

2.1 The displacement functions of the curved Timo–shenko beam element for the in-plane vibrations

The curved beam element in Figure 1 is in equilibrium under the loads shown, and a force is applied to it in the tangential direction. If the tangential forces acting on the curved beam are large, the vibrational behavior of the beam changes. In this analysis, the effect of tan–gential force, known as geometric stiffness, is not con–sidered. The cutting angle or ψ, as shown in Figure 1, is measured relative to the line perpendicular to the midplane of the beam, and its positive direction is coun–terclockwise.

The cutting angle is calculated from (1).

$$\psi = \frac{F_V}{kGA} \tag{1}$$

The rotation of the beam cross-section γ is obtained from (2).

$$\gamma = \frac{1}{r_0}\left(\frac{dv}{d\theta} - u\right) + \psi \tag{2}$$

where $r_0 = \dfrac{A}{\int_{r_a}^{r_b} (b/r)\,dr}$. It is a radius that there is no tension is created under the shear bending load with this radius (how to obtain the relation is given in Appendix A).

By considering small displacements for the element, it can be shown that the moment in this element is obtained from (3) (see Appendix A).

$$M = C_1\left(\frac{d^2v}{d\theta^2} - \frac{du}{d\theta} + C_2\frac{dF_v}{d\theta}\right) \tag{3}$$

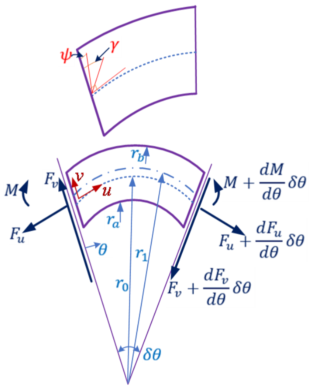

Figure 1. Forces and displacements in the curved beam element

The stress-strain relationship in the tangential direction of the beam element is calculated by (4).

$$F_u = \frac{EA}{r_0}\left(\frac{du}{d\theta} + v\right) \tag{4}$$

The three static equilibrium relations for this element are written as (5) to (7).

$$\frac{dM}{d\theta} = r_0 F_v \tag{5}$$

$$\frac{dF_u}{d\theta} = F_v \tag{6}$$

$$\frac{dF_v}{d\theta} = -F_u\,. \tag{7}$$

Equations (3) to (7) are written for a thick beam, which means, in these equations, the effect of shear deformation is considered. These equations can also be used for thin beams by changing the coefficients C_1 and C_2 in (3), ($C_1 = \frac{EI}{r_1^2}$, $C_2 = 0$).

In thin beams, the neutral axis coincides with the central axis of the beam cross-section (That is, (4) to (7), $r_0 = r_1$).

The complete solution of (3) to (7) is in the form of (8) to (11) (see Appendix B).

$$\{u\} = [H]\{G_i\} \tag{9}$$

$$\{f\} = [d]\{G_i\} \tag{10}$$

$$\begin{Bmatrix} F_u \\ F_v \\ M \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_7\cos\theta & C_7\sin\theta \\ 0 & 0 & 0 & 0 & -C_7\sin\theta & C_7\cos\theta \\ 0 & C_1 & 0 & 0 & C_6\cos\theta & C_6\sin\theta \end{bmatrix} \begin{Bmatrix} G_1 \\ \vdots \\ G_6 \end{Bmatrix} \tag{11}$$

where $\{G_i\}^T = \{G_1 \;\; \ldots \;\; G_6\}$ are the arbitrary integral constants, which are determined by the boundary conditions of the curved beam element. In Equation (8), γ which is the rotation of the beam cross-section, it is given for the simplicity of work along with solving the displacements. The definition of all constants of beam elements C_i is given in Appendix C.

2.2 Shape functions for Timoshenko curved beam element at in-plane vibrations

Equations (9) and (10), which are the solution of equu–ilibrium and stress-strain equations, they can be used to obtain the stiffness matrix of the curved beam element.

The boundary conditions for the displacement of the curved beam element shown in Figure 2 are applied to (9), and (12) is obtained.

$$\{\delta\} = [B]\{G_i\} \tag{12}$$

$$\begin{Bmatrix} u \\ v \\ \gamma \end{Bmatrix} = \begin{bmatrix} 1 & -\theta & \cos\theta & -\sin\theta & \theta\cos\theta & \theta\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta & (\theta\sin\theta + C_4\cos\theta) & (C_4\sin\theta - \theta\cos\theta) \\ -1/r_0 & \theta/r_0 & 0 & 0 & C_5\sin\theta/r_0 & -C_5\cos\theta/r_0 \end{bmatrix} \begin{Bmatrix} G_1 \\ \vdots \\ G_6 \end{Bmatrix}. \tag{8}$$

$$\begin{Bmatrix} u_1 \\ v_1 \\ \gamma_1 \\ u_2 \\ v_2 \\ \gamma_2 \end{Bmatrix} = \begin{bmatrix} 1 & -\theta_1 & \cos\theta_1 & -\sin\theta_1 & \theta_1\cos\theta_1 & \theta_1\sin\theta_1 \\ 0 & 1 & \sin\theta_1 & \cos\theta_1 & (\theta_1\sin\theta_1 + C_4\cos\theta_1) & (C_4\sin\theta_1 - \theta_1\cos\theta_1) \\ -1/r_0 & \theta_1/r_0 & 0 & 0 & C_5\sin\theta_1/r_0 & -C_5\cos\theta_1/r_0 \\ 1 & -\theta_2 & \cos\theta_2 & -\sin\theta_2 & \theta_2\cos\theta_2 & \theta_2\sin\theta_2 \\ 0 & 1 & \sin\theta_2 & \cos\theta_2 & (\theta_2\sin\theta_2 + C_4\cos\theta_2) & (C_4\sin\theta_2 - \theta_2\cos\theta_2) \\ -1/r_0 & \theta_2/r_0 & 0 & 0 & C_5\sin\theta_2/r_0 & -C_5\cos\theta_2/r_0 \end{bmatrix} \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \end{Bmatrix}. \tag{13}$$

$$T = 0.5\rho\omega^2\{G_i\}^T\int_{\theta_1}^{\theta_2}(C_8[R_1R_1^T]+C_8[R_2R_2^T]+C_9([R_3R_1^T]+[R_1R_3^T])+C_{10}[R_3R_3^T])\{G_i\}\,d\theta \quad (21)$$

$$R_2 = \{0 \quad 1 \quad \sin\theta \quad \cos\theta \quad (\theta\sin\theta + C_4\cos\theta) \quad (C_4\sin\theta - \theta\cos\theta)\}^T \quad (23)$$

$$[H] = \int_{\theta_1}^{\theta_2}\left\{C_8R_1R_1^T + C_8R_2R_2^T + C_9(R_3R_1^T + R_1R_3^T) + C_{10}R_3R_3^T\right\}d\theta \quad (26)$$

If $\{G_i\}$ is calculated using (12) and substitute in (9), the shape functions of the element are obtained as (14).

$$\{u\} = [H][B]^{-1}\{\delta\} = [N]\{\delta\} \quad (14)$$

Therefore, according to the definition of shape func–tions, it can be written.

$$[N] = [H][B]^{-1} \quad (15)$$

For simplicity, shape functions are implicitly obtained, and their explicit writing has been avoided.

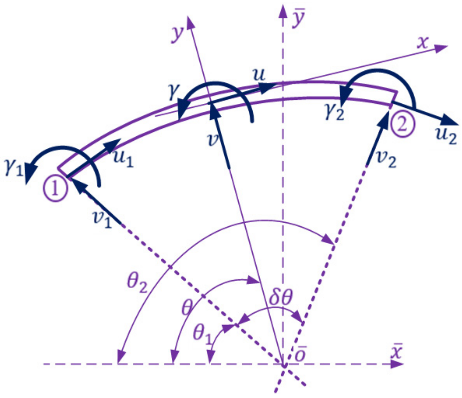

Figure 2. Definition of in-plane displacements for the curved beam element

2.3 The stiffness matrix of the Timoshenko curved beam element

The static equilibrium between the forces shown in nodes 1 and 2 for the curved beam element in Figure 1 can be written as (16).

$$\{F_{u1} \quad F_{v1} \quad M_1\} = -\{F_{u2} \quad F_{v2} \quad M_2\}. \quad (16)$$

By using (10) about nodes 1 and 2 of Figure 1 and by pay attention to (16) can be written as:

$$\{F\} = [D]\{G_i\}. \quad (17)$$

$$\begin{Bmatrix} F_{u1} \\ F_{v1} \\ M_1 \\ F_{u2} \\ F_{v2} \\ M_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -C_7\cos\theta_1 & -C_7\sin\theta_1 \\ 0 & 0 & 0 & 0 & -C_7\sin\theta_1 & C_7\cos\theta_1 \\ 0 & -C_1 & 0 & 0 & -C_6\cos\theta_1 & -C_6\sin\theta_1 \\ 0 & 0 & 0 & 0 & C_7\cos\theta_2 & C_7\sin\theta_2 \\ 0 & 0 & 0 & 0 & C_7\sin\theta_2 & -C_7\cos\theta_2 \\ 0 & C_1 & 0 & 0 & C_6\cos\theta_2 & C_6\sin\theta_2 \end{bmatrix} \begin{Bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \end{Bmatrix} \quad (18)$$

It is sufficient to obtain $\{G_i\}$ from (12) and replace it in (17) to obtain the stiffness matrix of the curved beam element.

$$\{F\} = [D]\{G_i\} = [D][B]^{-1}\{\delta\} = [K]\{\delta\}. \quad (19)$$

where $[K] = [D][B]^{-1}$ is the stiffness matrix of the curved beam element in in-plane vibrations. This matrix is symmetric. This property of the stiffness matrix accommodates a suitable way of checking the correctness of the analysis.

2.4 The mass matrix of the Timoshenko curved beam element

A general term for the kinetic energy of a curved beam element that vibrates with frequency ω on its plane is in the form of:

$$T = 0.5\rho\omega^2\int_{\theta_1}^{\theta_2}\int_{r_a}^{r_b} br\left\{v^2 + \left[u + (r_0 - r)\gamma\right]^2\right\}dr\,d\theta \quad (20)$$

If (20) is written in matrix form, and the inner integral is calculated, (21) is obtained:

where

$$R_1 = \{1 \quad -\theta \quad \cos\theta \quad -\sin\theta \quad \theta\cos\theta \quad \theta\sin\theta\}^T \quad (22)$$

$$R_3 = \{-1/r_0 \quad \theta/r_0 \quad 0 \quad 0 \quad C_5\sin\theta/r_0 \quad -C_5\cos\theta/r_0\}^T \quad (24)$$

In order to obtain the kinetic energy relation in thin beams, it is sufficient to apply changes $C_9 = C_{10} = 0$, $r_0 = r_1$ in (21). By substituting (12) in (21), the vector $\{G_i\}$ is removed, and (21) is simplified to:

$$T = 0.5\rho\omega^2\{\delta\}^T[B]^{-T}[H][B]^{-1}\{\delta\} \quad (25)$$

where the mass matrix of a curved beam element $\rho[B]^{-T}[H][B]^{-1}$ can be easily obtained.

2.5 Vibration absorbing element

If the vibration absorber (Figure 3) is considered an element with two nodes, the stiffness, mass, and damping matrices of the vibration absorber element are in the form of (27) to (29) [36].

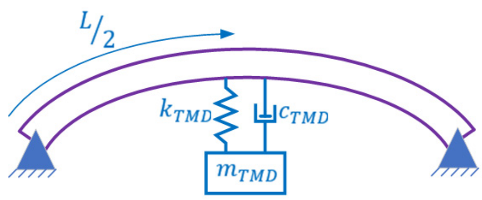

Figure 3. The curved beam with a vibration absorber

$$[m_{TMD}] = \left(\frac{1}{\rho R}\right)\begin{bmatrix} 0 & 0 \\ 0 & m_{TMD} \end{bmatrix} \quad (27)$$

$$[K_{TMD}] = \left(\frac{R^2}{EI_x}\right)\begin{bmatrix} K_{TMD} & -K_{TMD} \\ -K_{TMD} & K_{TMD} \end{bmatrix} \quad (28)$$

$$[C_{TMD}] = \begin{bmatrix} C_{TMD} & -C_{TMD} \\ -C_{TMD} & C_{TMD} \end{bmatrix} \quad (29)$$

It should be noted that for obtaining the optimal answer, the range of dimensionless parameters of vibration absorber must be observed. This range is as follows:

$$f_{TMD} = \frac{\omega_{TMD}}{\omega_n}, \; 0 \le f_{TMD} \le 1 \quad (30)$$

$$\xi_{TMD} = \frac{C_{TMD}}{2\sqrt{K_{TMD} m_{TMD}}}, \; 0 \le \xi_{TMD} \le 2.5 \quad (31)$$

3. NUMERICAL EXAMPLES AND DISCUSSIONS

2.6 Example 1: A curved beam with simply supported boundary conditions

The first example calculates the natural frequencies of a curved beam with and without absorber and simply supported boundary conditions (S-S). The dimensions and material properties have been selected from the reference [34,36] and shown in Table 1.

The first five frequencies of this beam with S-S boundary conditions are given in Table 2. These frequencies have been compared with the values given in the references [33,34].

The natural frequencies in the second column of Table 2 are obtained using the exact solution of the frequency equations in the reference [34]. In reference [33] (fourth column of Table 2), natural frequencies are calculated using the straight beam element. It is shown that the exact answer was not obtained using 40 elements.

By comparing the results obtained from the In-plane computer program and the results are given in the references [33,34], it can be seen that by using the same number of curved elements, the numerical values of the frequencies obtained from this program are less than the frequencies given in the reference [33].

Table 1. Geometrical and physical data of the curved beam

Parameter	value
Radial thickness of the beam [cm]	0.0327
Axial thickness of the beam [cm]	2.560
Radius of curvature [cm]	R = 76.200
Beam cross-sectional area [cm^2]	0.0839
shear correction factor	k = 0.800
The central angle of the curved beam [rad]	$\alpha = 1$
$\rho[\frac{kg}{m^3}]$	2764
$E\,[\frac{N}{m^2}]$	6.89×10^{10}
υ[-]	0.300
M_{TMD}[kg]	100000
K_{TMD}[N / m]	475000.320
C_T[N.s / m]	63547.970

This is due to the fact that in the reference [33], only the rotational inertia of the curved beam is considered while in this program, in addition to the rotational inertia, the effects of shear deformation are also considered. As a result, the overall stiffness, and its natural frequencies will be reduced.

By comparing the second and 11th columns in Table 2, it can be seen that using 20 elements, the answers obtained from the computer program have been reduced from the reference answers [34].

It should be noted that the reference [34] does not take into account the rotational inertia of the beam, and therefore at high frequencies the answers obtained from the computer program are less than the values given in the reference [34].

Table 2. Natural frequencies of curved beams (Hz) with S-S boundary conditions ($v_{Left} = v_{Right} = 0$).

Mode number	exact solution [34]	Reference [33]					In-plane code without absorber				In-plane code with absorber			
		With straight beam elements		With curved beam elements			With curved beam elements				With curved beam elements			
		20 ELE	40 ELE	2 ELE	4 ELE	6 ELE	2 ELE	4 ELE	6 ELE	20 ELE	2 ELE	4 ELE	6 ELE	8 ELE
1	0.349	0.565	0.564	0.349	0.349	0.349	0.349	0.349	0.349	0.349	0.349	0.349	0.349	0.349
2	1.571	1.952	1.952	1.572	1.572	1.572	1.572	1.571	1.571	1.571	1.534	1.534	1.534	1.534
3	3.612	4.228	4.226	3.725	3.615	3.613	3.723	3.614	3.611	3.611	3.723	3.614	3.611	3.610
4	6.470	7.230	7.230	8.212	6.474	6.474	8.207	6.471	6.471	6.468	8.151	6.432	6.432	6.424
5	10.144	11.161	11.156	14.307	10.274	10.162	14.299	10.268	10.157	10.123	14.299	10.268	10.157	10.142

2.7 Example 2

The second example is considered a curved beam with two different boundary conditions: clamped-clamped (C-C) and simply supported (S-S) conditions. The material properties and dimensions of the beam are selected from reference [33,36] and shown in Table 3.

Table 3. Geometrical and physical data of the curved beam.

Parameter	value
Radial thickness of the beam [m]	0.8
Axial thickness of the beam [m]	1.5
Radius of curvature [m]	R = 20
shear correction factor	k = 0.800
The central angle of the curved beam	$\alpha = 120^{\circ}$
$\rho[\frac{kg}{m^3}]$	2700
$E\ [\frac{N}{m^2}]$	12×10^{10}
υ[-]	0.300
M_{TMD}[kg]	100000
K_{TMD}[N / m]	475000.320
C_T[N.s / m]	63547.970

The first five frequencies of the curved beam are calculated with C-C and S-S boundary conditions and have been shown in Table 4. In the computer program 40 elements have been used to calculate the natural frequencies of a curved beam. By comparing the numbers in Table 4, it can be seen that the frequencies calculated by the computer program are still slightly lower than the values given in the reference [33] (the reason mentioned).

There is a small difference between the frequencies of Timoshenko (In-plane program) and Euler-Bernoulli (reference). The reason for this slight difference can be explained by calculating the slenderness coefficient of the example. $S_r = \frac{b^2}{R^2\alpha} = \frac{1.5^2}{20^2(2\pi/3)} = 0.00269$ As can be seen, the slenderness coefficient of the beam is less than 0.01. Therefore, the effects of shear forces can be igno–red.

It is observed that the frequencies of the curved beam with the absorber are lower than without the absorber. The reason is the changes in the stiffness and total mass matrices.

The mode shape of the C-C beam is plotted without and with absorber in Figure 4 and Figure 5, respectively.

Figure 6 shows the displacement of the nodal points of a curved beam with C-C boundary conditions, and without absorber, for the first four modes of this beam.

Figure 7 shows the displacement of the nodal points of a curved beam with C-C boundary conditions and with absorber for the first four modes of this beam.

According to Figure 6, it can be seen that v has more values than u and γ in all modes. That is, the displacement values related to the y direction are greater than the rotation values of the cross-section and the displacement in the x direction (Figure 7).

The mode shape of the S-S beam is plotted without and with absorber in Figure 8 and 9, respectively.

Figure 10 and 11 shows the displacement of the nodal points of a curved beam with S-S boundary conditions, and without and with absorber, for the first four modes of this beam, respectively.

According to Figure 10 and 11, it can be seen that v has more values than u and γ in all modes. That is, the displacement values related to the y direction are greater than the rotation values of the cross-section and the displacement in the x direction.

2.8 Example 3

In the third example, the beam radius has been altered between 2 to 10 m, and the first three frequencies of the curved beam are calculated. The rest of the parameters are the same as in the second example.

In Figure 12, the frequency calculated by a computer program (Timoshenko beam theory) has been compared by the results of ref. [33] (Euler-Bernoulli beam theory). For calculating the natural frequencies in the computer program, the curved beam is modeled with 20 elements.

It can be seen in Figure 12, the frequencies of the Timoshenko curved beam less than the frequencies of the Euler-Bernoulli beam. The reason is that the effects of shear deformation, which reduces the stiffness and the natural frequencies of the curved Timoshenko beam.

Moreover, it is observed that with the increasing curved beam slender coefficient, the difference between the frequencies calculated of the two theories has increased and cannot be ignored. In this case, the more accurate theory or the Timoshenko beam theory must be used.

It is also observed that in the second mode, there is not much difference between the frequencies of the curved beam in the theory of Timoshenko and Euler Bernoulli. This ignorable difference is because the second mode shape, which is called the breathing mode, has a more radial displacement, and the angular displacement is almost zero, and therefore the shear angle is equal to zero in this mode shape.

Table 4. Natural frequencies of curved beams (Hz) with C-C and S-S boundary conditions.

Mode number	C-C			S-S		
	Reference [33]	In-plane code without absorber	In-plane code with absorber	Reference [33]	In-plane code without absorber	In-plane code with absorber
1	27.892	27.746	0.1007	16.316	16.284	0.10066
2	54.993	54.507	27.7460	41.069	40.893	16.2844
3	100.710	99.418	54.5071	79.357	78.790	40.8933
4	143.866	141.721	99.4179	124.036	122.730	78.7902
5	198.849	198.392	141.7208	187.173	181.474	122.7296

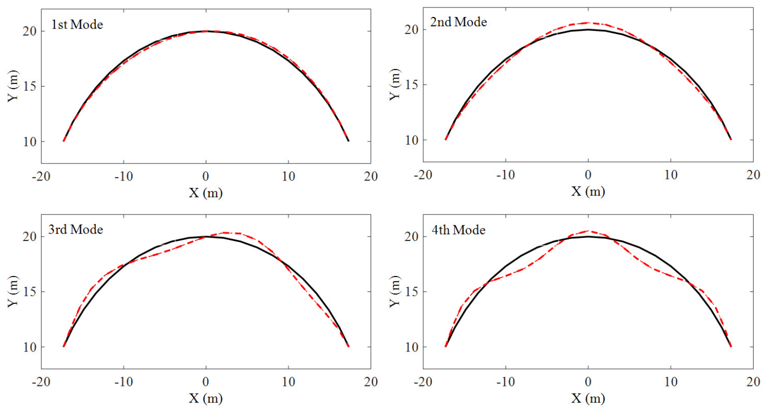

Figure 4. The mode shape of the curved beam without absorber for the first four modes (C-C)

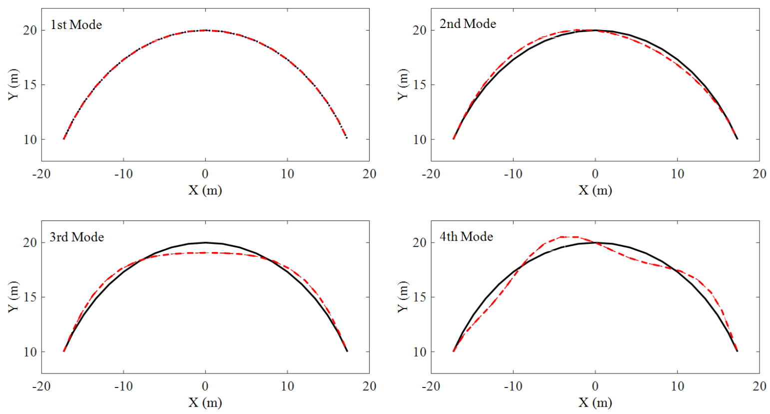

Figure 5. The mode shape of the curved beam with absorber for the first four modes (C-C)

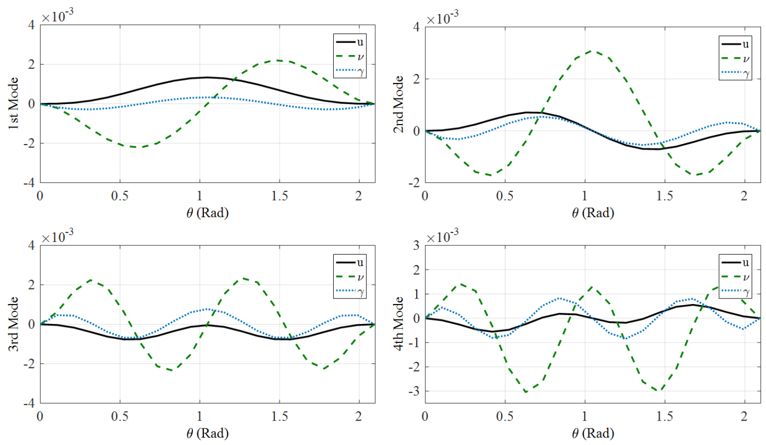

Figure 6. Displacement of nodal points of the curved beam without absorber for the first four modes (C-C)

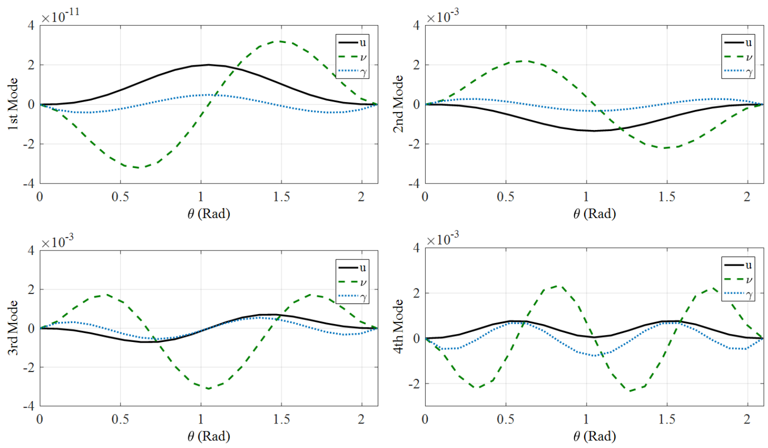

Figure 7. Displacement of nodal points of the curved beam with absorber for the first four modes (C-C)

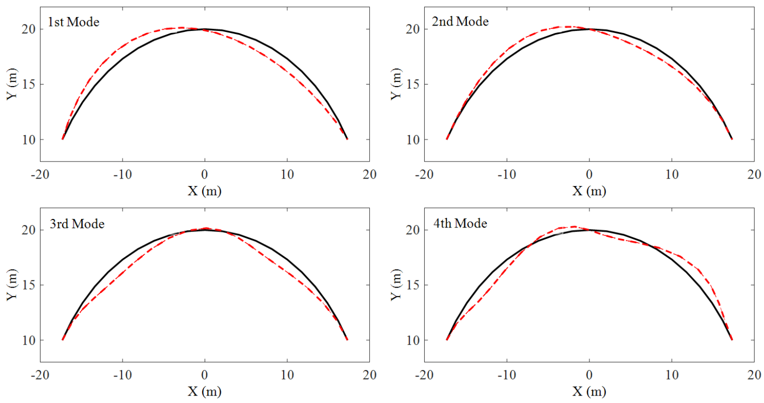

Figure 8. The mode shape of the curved beam without absorber for the first four modes (S-S)

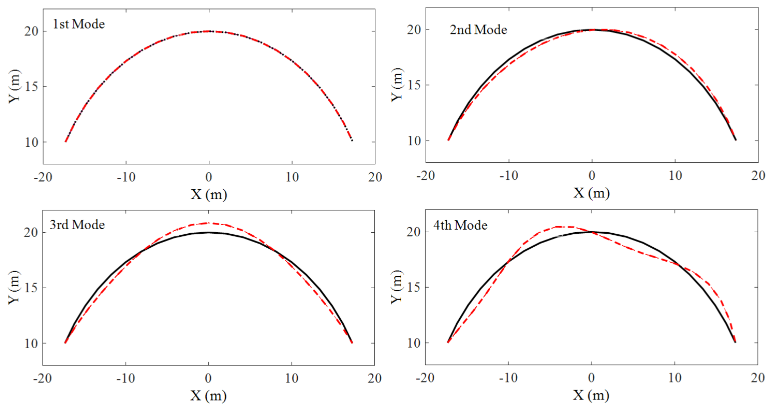

Figure 9. The mode shape of the curved beam with absorber for the first four modes (S-S)

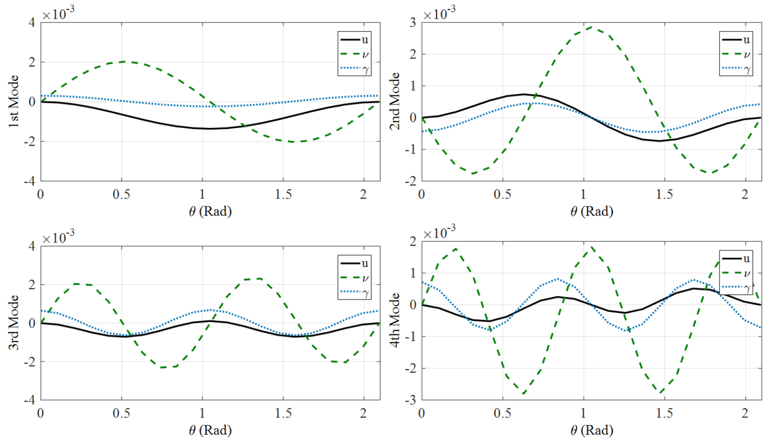

Figure 10. Displacement of nodal points of the curved beam without absorber for the first four modes (S-S)

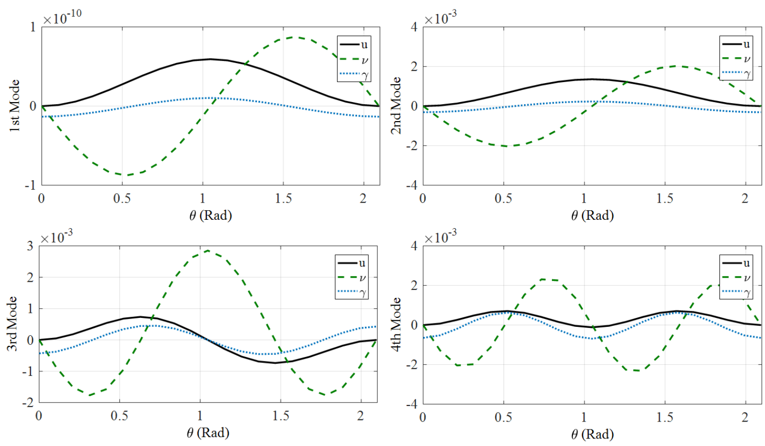

Figure 11. Displacement of nodal points of the curved beam with absorber for the first four modes (S-S)

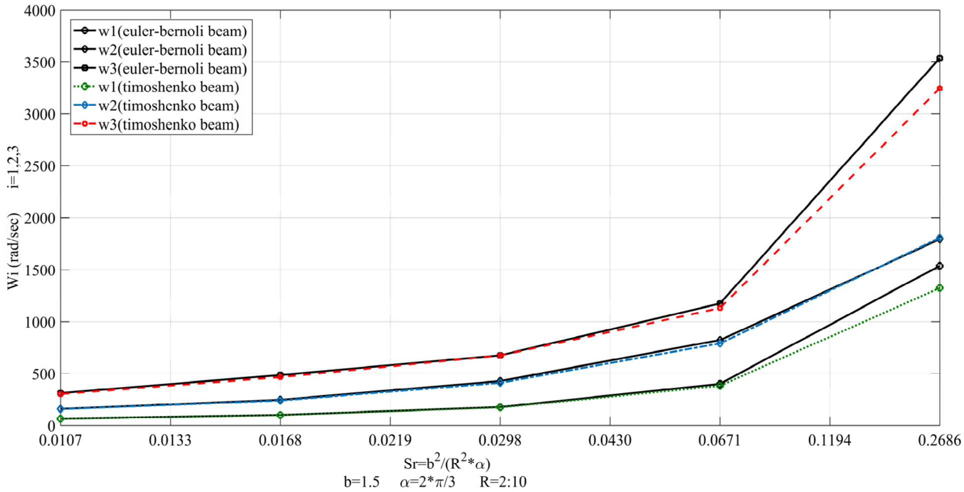

Figure 12. Natural frequencies of Timoshenko and Euler-Bernoulli curved beam with the S-S condition versus the slender coefficient

4. CONCLUSION

In order to analyze the in-plane vibrations of the curved Timoshenko beam, the mass and stiffness matrices of the curved beam were obtained using finite element methods for in-plane vibration mode. The stiffness and mass matrices of the curved beam were obtained from the force-displacement relationships and the kinetic energy equations, respectively. Due to the use of the local polar coordinate system, the elements were assembled easily without the need for coordinate conversion matrices. Curved beam frequencies with different boundary conditions were calculated using the Timoshenko theory, and the effect of the slender coefficient on the difference between Timoshenko and Euler-Bernoulli theory was investigated. Therefore, as the curvature of the curved beam increases, the difference between the frequencies calculated from the two theories of Timoshenko and Euler-Bernoulli increases, and this difference cannot be ignored. In these cases, a more accurate theory, namely the theory of the Timoshenko beam, must be used. The natural frequencies for the state with the vibration absorber are lower than without the vibration absorber, and the convergence of the finite element method was well after using 20 elements. Moreover, the maximum displacement value with the vibration absorber on the curved beam is less than without one, and this shows the efficiency of the vibration absorber in reducing the displacement value.

APPENDIX A: OBTAINING THE RADIUS OF THE NEUTRAL LINE AND THE STRESS-STRAIN RELATIONSHIP IN THE ELEMENT UNDER BENDING

This appendix describes how to obtain two relationships. First, the radius at which pure bending in the thick curved beam element does not create any stress is obtained, and then the stress-strain interface in the bending element is calculated.

The thick curved beam element, under pure bending moment, is shown in Figure A-1. The initial plate length of the beam that is in radius r relative to the center of the beam is equal to $r\delta\theta$. In the deformed state from the shape geometry, increasing the length of this element is equal to (A-1).

$$\frac{R(r_0 - r)}{R + r_0}\delta\theta . \tag{A-1}$$

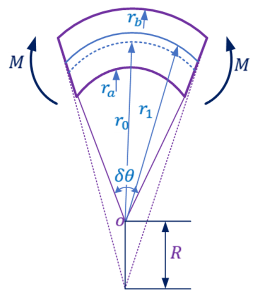

Figure A-1. Thick curved beam element, under a pure bending moment.

The tensile force δF at the end of this plate is obtained from (A-2).

$$\delta F = b\,\delta r \frac{R}{r}\left(\frac{r_0 - r}{R + r_0}\right) E . \tag{A-2}$$

The net tangential force is equal to the sum of the tensile forces on the cross-section of the beam.

$$\int_{r_a}^{r_b} \delta F = \int_{r_a}^{r_b} \frac{bR}{r}\left(\frac{r_0 - r}{R + r_0}\right) E\,dr . \tag{A-3}$$

For small displacements $R << r_0, r_1$, and therefore (A-4) is obtained.

$$\frac{ER}{r_0}\int_{r_a}^{r_b} b(r_0 / r - 1)\,dr = ER\int_{r_a}^{r_b} \frac{b}{r}dr - \frac{R}{r_0}AE . \tag{A-4}$$

In the case of pure bending, the net tangential force must be zero, so (A-5) is obtained for the neutral axis radius.

$$r_0 = \frac{A}{\int_{r_a}^{r_b} (b/r)\,dr} . \tag{A-5}$$

The bending moment around the center of curvature of the curved beam element (point o in Figure A-1) is equal to:

$$M = -\int_{r_a}^{r_b} r\,\delta F = -\int_{r_a}^{r_b} bR\left(\frac{r_0 - r}{R + r_0}\right) E\,dr . \quad \text{(A-6)}$$

Assuming small displacements, (A-6) takes the form (A-7).

$$M = -\frac{ER}{r_0}\int_{r_a}^{r_b} b\,(r_0 - r)\,dr . \quad \text{(A-7)}$$

Now with respect to the two (A-8) and (A-9).

$$\int_{r_a}^{r_b} b\,r\,dr = r_1 A . \quad \text{(A-8)}$$

$$\int_{r_a}^{r_b} b\,r_0\,dr = r_0 A . \quad \text{(A-9)}$$

A simplified (A-10) is obtained.

$$M = ERA\left(\frac{r_1 - r_0}{r_0}\right) . \quad \text{(A-10)}$$

On the other hand, due to the shape geometry.

$$R = r_0 \frac{d\gamma}{d\theta} . \quad \text{(A-11)}$$

where γ is the angle of rotation of the beam cross-section, therefore:

$$R = r_0 \frac{d}{d\theta}\left(\frac{1}{r_0}\frac{dv}{d\theta} - \frac{u}{r_0} + \psi\right) . \quad \text{(A-12)}$$

According to (A-11) and (A-12) are.

$$M = EA\left(\frac{r_1 - r_0}{r_0}\right)\left(\frac{d^2 v}{d\theta^2} - \frac{du}{d\theta} + r_0\frac{d\psi}{d\theta}\right) . \quad \text{(A-13)}$$

$$M = C_1\left(\frac{d^2 v}{d\theta^2} - \frac{du}{d\theta} + C_2\frac{dF_v}{d\theta}\right) . \quad \text{(A-14)}$$

Equation (A-14) is the stress-strain relationship in which the thick curved beam element is subjected to pure bending load on its plane.

APPENDIX B: SOLVING DIFFERENTIAL EQUATIONS (3) TO (7)

This appendix describes how to solve differential (3) to (7).

$$M = C_1\left(\frac{d^2 v}{d\theta^2} - \frac{du}{d\theta} + C_2\frac{dF_v}{d\theta}\right) . \quad \text{(B-1)}$$

$$F_u = C_3\left(\frac{du}{d\theta} + v\right) . \quad \text{(B-2)}$$

$$\frac{dM}{d\theta} = r_0 F_v . \quad \text{(B-3)}$$

$$\frac{dF_u}{d\theta} = F_v . \quad \text{(B-4)}$$

$$\frac{dF_v}{d\theta} = -F_u . \quad \text{(B-5)}$$

It is obtained (B-6) by using (B-1), (B-2) and (B-5).

$$\begin{aligned} & M = C_1[v'' - u' - C_2C_3(u' + v)] \\ & \to M' = C_1[v''' - u'' - C_2C_3(u'' + v')] \end{aligned} . \quad \text{(B-6)}$$

(B-7) is obtained from (B-2) to (B-4).

$$M' = r_0 C_3 (u'' + v') . \quad \text{(B-7)}$$

According to (B-6) and (B-7) are:

$$\begin{aligned} & -C_1 u'' - C_1C_2C_3 u'' - r_0C_3 u'' \\ & \quad = -C_1 v''' + C_1C_2C_3 v' + r_0 C_3 v' \end{aligned} . \quad \text{(B-8)}$$

Then by using (B-2) and (B-4).

$$\begin{aligned} & C_3(u'' + v') = F_v \xrightarrow{\text{der.}\&6} \\ & C_3(u''' + v'') = -C_3(u' + v) \to u^{(4)} + u'' = -v''' - v' \end{aligned} . \quad \text{(B-9)}$$

By removing v between (B-8) and (B-9), (B-10) is obtained.

$$u^{(6)} + 2u^{(4)} + u'' = 0 . \quad \text{(B-10)}$$

which is a 6th order differential equation, and its solution is in the form of (B-11).

$$\begin{aligned} u = {} & G_1 - G_2\theta + G_3\cos\theta - G_4\sin\theta \\ & + G_5\theta\cos\theta + G_6\theta\sin\theta \end{aligned} . \quad \text{(B-11)}$$

v can also be obtained by replacing u in (B-8).

$$\begin{aligned} v = {} & G_2 + G_3\sin\theta + G_4\cos\theta \\ & + G_5(\theta\sin\theta + C_4\cos\theta) + G_6(C_4\sin\theta - \theta\cos\theta) \end{aligned} . \text{(B-12)}$$

By obtaining u , v and with the given relations, other unknowns can be easily obtained. By replacing u and v in (B-2), (B-13) is obtained.

$$\begin{aligned} F_u = {} & C_3[-G_2 - G_3\sin\theta - G_4\cos\theta + G_5(\cos\theta - \theta\sin\theta) \\ & + G_6(\sin\theta + \theta\cos\theta) + G_2 + G_3\sin\theta + G_4\cos\theta \\ & + G_5(\theta\sin\theta + C_4\cos\theta) + G_6(C_4\sin\theta - \theta\cos\theta)] \\ = {} & C_3[G_5(1 + C_4)\cos\theta + G_6(1 + C_4)\sin\theta] \\ = {} & C_7\cos\theta\, G_5 + C_7\sin\theta\, G_6 \end{aligned} . \quad \text{(B-13)}$$

With the help of (B-4), (B-14) is obtained.

$$F_v = -C_7\sin\theta\, G_5 + C_7\cos\theta\, G_6 . \quad \text{(B-14)}$$

Substituting u , v and F_v in (B-1) and simplification, M is also obtained.

APPENDIX C: DEFINITION OF CONSTANTS $C_1 - C_{10}$

Mass and stiffness matrices are defined by (C-1) and (C-2) relations.

$$[K] = [D][B]^{-1} . \quad \text{(C-1)}$$

$$[M] = \rho[B]^{-T}[H][B]^{-1} . \quad \text{(C-2)}$$

Constants C_1 to C_{10} are defined as (C-3) to (C-12) in the curved beam element.

$$C_1 = EA\left(\frac{r_1 - r_0}{r_0}\right) . \quad \text{(C-3)}$$

$$C_2 = \frac{r_0}{kGA} . \tag{C-4}$$

$$C_3 = \frac{EA}{r_0} . \tag{C-5}$$

$$C_4 = \frac{r_0 C_3 - C_1(1 - C_2 C_3)}{-r_0 C_3 - C_1(1 + C_2 C_3)} . \tag{C-6}$$

$$C_5 = 1 - C_4 - C_2 C_3 (1 + C_4) . \tag{C-7}$$

$$C_6 = C_1 C_5 . \tag{C-8}$$

$$C_7 = C_3 (1 + C_4) . \tag{C-9}$$

$$C_8 = \int_{r_a}^{r_b} b\, r\, dr . \tag{C-10}$$

$$C_9 = \int_{r_a}^{r_b} (r_0 - r)\, b\, r\, dr . \tag{C-11}$$

$$C_{10} = \int_{r_a}^{r_b} (r_0 - r)^2\, b\, r\, dr . \tag{C-12}$$

where

$$r_0 = \frac{A}{\int_{r_a}^{r_b} (b / r)\, dr} . \tag{C-13}$$

For a beam with a rectangular cross-section, the relations are calculated as (C-14) to (C-17).

$$r_0 = h \Big/ \log\left(\frac{1 + h / 2r_l}{1 - h / 2r_l} \right) . \tag{C-14}$$

$$C_8 = A r_l . \tag{C-15}$$

$$C_9 = A r_l (r_0 - r_l - I / A r_l) . \tag{C-16}$$

$$C_{10} = A r_l \left\lfloor r_0^2 + r_l^2 + h^2 / 4 - 2 r_0 (I / A r_l + r_l) \right\rfloor . \tag{C-17}$$

In order to rewrite (C-14) to (C-17) for thin curved beams, in which there are no rotational shear angle and inertia effects. The constants must be defined as (C-18) to (C-20).

$$C_1 = EI / r_l^2 . \tag{C-18}$$

$$C_2 = C_9 = C_{10} = 0 . \tag{C-19}$$

$$r_0 = r_l . \tag{C-20}$$

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МОДЕЛ КОНАЧНИХ ЕЛЕМЕНАТА КРУЖНО ЗАКРИВЉЕНЕ ТИМОШЕНКОВЕ ГРЕДЕ ЗА АНАЛИЗУ ВИБРАЦИЈА У РАВНИ

А. Нади, М. Рагхеби

Закривљене греде се највише користе код лукова и железничких мостова као и код опреме за забавне паркове. Има мало радова о утицају закривљене греде на деформацију смицања и ротациону инер–цију. Рад истражује примену новог модела коначних елемената за анализу вибрација у равни код Тимо-шенкове закривљене греде. Матрице крутости и масе елемената греде добијене су из односа сила-померај односно једначина за кинетику енергију. Склоп матрица елементарних својстава је једнос-таван без потребе за матрицом трансформације јер се користи локални поларни координатни систем. Природне фреквенције закривљене Ојлер-Бернули-јеве греде велике дебљине нису довољно прецизне. У овом случају, потребно је користити Тимошен-кову греду. Разматра се утицај апсорбера вибрација на природне фреквенције закривљене греде.