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# Method of Strength Analysis for Doubly-Curved Stiffened Orthotropic Shells by Various Strength Theories

*The paper proposes a method of strength analysis for materials of thin-walled shell structures reinforced with stiffeners. The shells under consideration were made of orthotropic materials. The authors analyze the applicability of the following seven strength criteria: the maximum stress criterion, the Mises–Hill criterion, the Fisher criterion, the Goldenblatt–Kopnov criterion, the Liu–Huang–Stout criterion, the Tsai–Wu criterion, and the Hoffman criterion. During the study, doubly-curved shallow shells square in the plan were considered. A geometrically nonlinear mathematical model of shell deformation, which considers transverse shears, was used. The calculations were based on the characteristics of T-10/UPE22-27 glass-fiber-reinforced plastic. The method relies on calculating the values of several strength criteria at each step of structural loading and analyzing the development of areas failing to meet the strength conditions as the load increases.*

***Keywords:*** *shells, strength criteria, buckling, maximum stresses, stiffened shell.*

## 1. INTRODUCTION

The study of the process of deformation of shell struc–tures is essential for various industries, including air–craft building, shipbuilding, rocket science, and others. In construction, such structures are often used, among other things, for covering large-span structures, for example, stadiums, concert halls, markets, warehouses, hangars for machinery and equipment, and factory buildings.

Thin-walled shell structures may lose their perfor–mance due to buckling when a small change in the load results in a significant rapid increase in displacements (deflection) and irreversible changes in the material (loss of strength).

Unfortunately, most available studies analyze either buckling resistance or strength. However, in some cases, it is hard to predict what will be compromised earlier.

Thus, it needs to perform structural analysis in terms of buckling and occurrence of failure to meet the strength conditions (including using various strength theories).

Structures subjected to external loading can be ana–lyzed based on the limited state of structural materials [1, 2].

Here, loss of strength means a state when the material experiences irreversible transformations. In fact, loss of strength is determined when at least one point of the structure fails to meet the strength criterion.

With a further increase in loading, the areas failing to meet the strength conditions begin to expand, and it becomes important to analyze their distribution and development.

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Much attention is paid to the study of the deformation of composite materials, which are often orthotropic. So, in the work of Smerdov [3] for the basic classes of composites (high-, medium-, and low-mo–dulus carbon-fiber, organo-, and glass-fiber plastics), recommendations for a rational choice of their structures to obtain experimental results allowing one to identify the elastic characteristics of unidirectional composites are formulated.

Analyzing strength criteria applicable to orthotropic materials indicates the need for a generally accepted criterion. Consequently, it needs to use several criteria and ensure that the results obtained will be analyzed later. The results of such comparisons can be found in papers [4–8].

The studies addressing the strength of materials and structures often use the following criteria: the maximum stress criterion [4, 6, 9, 10], the Goldenblatt–Kopnov criterion [7, 11], the Tsai–Wu criterion [4, 12–16], the Hoffman criterion [4, 6, 17], and the Hashin criterion [13, 18–20].

Among the recent papers on the application and development of strength theories, the following can be mentioned: papers by Abrosimov and Elesin [6], Baryshev and Tsvetkov [21], Kolupaev et al. [22], Tsvetkov and Kulish [23], and Yu [24].

In [25], Korsun et al. describe in detail the key relations of the following strength criteria applicable to concrete: the Geniev criterion, the Geniev–Alikova criterion, the Leites criterion, the Yashin criterion, the Klovanich–Bezushko criterion, the Willam–Warnke criterion, and the Karpenko criterion.

The number of studies addressing the application of various strength theories to the analysis of shell struc–tures [4, 6, 26–31] is relatively small.

A valuable extensive comparative analysis of stren–gth criteria can be found in a paper by Oreshko et al. [8]. The researchers analyzed strength criteria appli–

cable to isotropic, orthotropic, and anisotropic materials. They also described the approaches used during the strength analysis of fibrous and layered composite materials. In the course of their study, they considered the following criteria: von Mises criterion, the Pisa–renko–Lebedev criterion, the William–Warnke criterion, the Drucker–Prager criterion, the Bazant criterion, the Norris criterion, the Cuntze criterion, the Goldenblatt–Kopnov criterion, the Tsai–Hill criterion, the Tsai–Wu criterion, the Hashin criterion, the LaRC criterion, the Hoffman criterion, the Puck criterion, the sandwich panel strength criteria, and others. Besides, they reviewed the strength models of the materials used in the ANSYS Mechanical APDL program.

The goal of this work is to present a method to analyze the strength of thin-walled shell structures made of orthotropic materials (including the use of various strength theories).

## 2. THEORY AND METHODS

### 2.1 Limit state criteria for orthotropic materials

As Tsvetkov and Kulish [23] noted, the phenome–nological criterion of strength in anisotropic materials relates the possibility of structural failure to the value of stress tensor ($\sigma_{ij}$) in the material and generally can be represented as follows: $f(\sigma_{ij}) \le 1$.

The expression includes a set of constants charac–terizing the structural behavior of the material. The strength criterion corresponds to the strength surface for a general case in the six-dimensional stress space $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{13}, \sigma_{23}, \sigma_{12}$.

The strength surface shall pass through the points determined by the technical strength characteristics of the material, i.e., ultimate strength under uniaxial tension and compression in three mutually perpen–dicular directions and ultimate strength in shear in three mutually perpendicular planes.

Considers the Liu–Huang–Stout criterion (in nota–tions accepted in the paper (Liu et al., [32])), which is the generalization of the Mises–Hill criterion (taking into account orthotropy but neglecting the difference between tensile and compressive moduli of elasticity) and the Drucker–Prager criterion (taking into account the difference between tensile and compressive moduli of elasticity but neglecting orthotropy).

$$\begin{gathered}\left\{F(\sigma_{22}-\sigma_{33})^2+G(\sigma_{33}-\sigma_{11})^2+H(\sigma_{11}-\sigma_{22})^2+\right.\\ \left.+2L\sigma_{23}^2+2M\sigma_{31}^2+2N\sigma_{12}^2\right\}^{\frac{1}{2}}+\\ +I\sigma_{11}+J\sigma_{22}+K\sigma_{33}\le 1,\end{gathered} \quad (1)$$

where

$$F=\frac{1}{2}\left\{\left(\frac{\sigma_2^T+\sigma_2^C}{2\sigma_2^T\sigma_2^C}\right)^2+\left(\frac{\sigma_3^T+\sigma_3^C}{2\sigma_3^T\sigma_3^C}\right)^2-\left(\frac{\sigma_1^T+\sigma_1^C}{2\sigma_1^T\sigma_1^C}\right)^2\right\},$$

$$G=\frac{1}{2}\left\{\left(\frac{\sigma_3^T+\sigma_3^C}{2\sigma_3^T\sigma_3^C}\right)^2+\left(\frac{\sigma_1^T+\sigma_1^C}{2\sigma_1^T\sigma_1^C}\right)^2-\left(\frac{\sigma_2^T+\sigma_2^C}{2\sigma_2^T\sigma_2^C}\right)^2\right\},$$

$$H=\frac{1}{2}\left\{\left(\frac{\sigma_1^T+\sigma_1^C}{2\sigma_1^T\sigma_1^C}\right)^2+\left(\frac{\sigma_2^T+\sigma_2^C}{2\sigma_2^T\sigma_2^C}\right)^2-\left(\frac{\sigma_3^T+\sigma_3^C}{2\sigma_3^T\sigma_3^C}\right)^2\right\},$$

$$I=-\frac{\sigma_1^T-\sigma_1^C}{2\sigma_1^T\sigma_1^C},\quad J=-\frac{\sigma_2^T-\sigma_2^C}{2\sigma_2^T\sigma_2^C},\quad K=-\frac{\sigma_3^T-\sigma_3^C}{2\sigma_3^T\sigma_3^C},$$

$$L=\frac{1}{2\left(\tau_{23}^S\right)^2},\quad M=\frac{1}{2\left(\tau_{31}^S\right)^2},\quad N=\frac{1}{2\left(\tau_{12}^S\right)^2}.$$

Here, $\sigma_1^T, \sigma_2^T, \sigma_3^T$ and $\sigma_1^C, \sigma_2^C, \sigma_3^C$ – ultimate strength under tension and compression in the $x, y, z$ directions, respectively; $\tau_{12}^S, \tau_{31}^S, \tau_{23}^S$ – ultimate strength in shear in the $xOy$, $xOz$, $yOz$ planes.

In a plane stress condition and considering that $\tau_{xz} = \tau_{yz} = 0$, the Liu–Huang–Stout criterion can be written as follows:

$$\begin{gathered}\left\{F\sigma_{22}^2+G\sigma_{11}^2+H(\sigma_{11}-\sigma_{22})^2+2N\sigma_{12}^2\right\}^{\frac{1}{2}}+\\ +I\sigma_{11}+J\sigma_{22}\le 1,\end{gathered} \quad (2)$$

where

$$F=\frac{1}{2}\left\{\left(\frac{\sigma_2^T+\sigma_2^C}{2\sigma_2^T\sigma_2^C}\right)^2-\left(\frac{\sigma_1^T+\sigma_1^C}{2\sigma_1^T\sigma_1^C}\right)^2\right\},$$

$$G=\frac{1}{2}\left\{\left(\frac{\sigma_1^T+\sigma_1^C}{2\sigma_1^T\sigma_1^C}\right)^2-\left(\frac{\sigma_2^T+\sigma_2^C}{2\sigma_2^T\sigma_2^C}\right)^2\right\},$$

$$H=\frac{1}{2}\left\{\left(\frac{\sigma_1^T+\sigma_1^C}{2\sigma_1^T\sigma_1^C}\right)^2+\left(\frac{\sigma_2^T+\sigma_2^C}{2\sigma_2^T\sigma_2^C}\right)^2\right\},$$

$$I=-\frac{\sigma_1^T-\sigma_1^C}{2\sigma_1^T\sigma_1^C},\quad J=-\frac{\sigma_2^T-\sigma_2^C}{2\sigma_2^T\sigma_2^C},\quad N=\frac{1}{2\left(\tau_{12}^S\right)^2}.$$

As Polilov and Tatus [11] noted, the strength criteria taking into account the directional fracture of fibrous composites make it possible to interpret experimental data and perform strength analysis of composite structural members more adequately.

To analyze orthotropic structures, seven criteria for the plane stress condition case, which in uniform notations can be represented as follows, are used:

Criterion 1 (the maximum stress criterion)

$$\frac{\sigma_{11}}{F_1^-}\le 1,\quad \frac{\sigma_{11}}{F_1^+}\le 1,\quad \frac{\sigma_{22}}{F_2^-}\le 1,\quad \frac{\sigma_{22}}{F_2^+}\le 1,\quad \left|\frac{\tau_{12}}{F_{12}}\right|\le 1. \quad (3)$$

In fact, it includes six criteria, thus providing the most information on the stress-strain state of the structure. However, it does not give any information on their cumulative effect or interaction.

Criterion 2 (the Mises–Hill criterion)

$$\frac{\sigma_{11}^2}{F_1^2}-\frac{\sigma_{11}\sigma_{22}}{F_1F_2}+\frac{\sigma_{22}^2}{F_2^2}+\frac{\tau_{12}^2}{F_{12}^2}\le 1. \quad (4)$$

Criterion 3 (the Fisher criterion (Fisher, [33]))

$$\frac{\sigma_{11}^2}{F_1^2} - K_f\frac{\sigma_{11}\sigma_{22}}{F_1F_2} + \frac{\sigma_{22}^2}{F_2^2} + \frac{\tau_{12}^2}{F_{12}^2} \le 1, \quad \text{where} \quad K_f = \frac{E_1(1+\mu_{21}) + E_2(1+\mu_{12})}{2\sqrt{E_1E_2(1+\mu_{21})(1+\mu_{12})}}. \tag{5}$$

Criterion 4 (the Goldenblatt–Kopnov criterion (Goldenblat and Kopnov, [34]))

$$\frac{1}{2}\left(\frac{1}{F_1^+} + \frac{1}{F_1^-}\right)\sigma_{11} + \frac{1}{2}\left(\frac{1}{F_2^+} + \frac{1}{F_2^-}\right)\sigma_{22} + \\ + \frac{1}{2}\left\{\left(\frac{F_1^+ - F_1^-}{F_1^+F_1^-}\right)^2\sigma_{11}^2 + \left(\frac{F_2^+ - F_2^-}{F_2^+F_2^-}\right)^2\sigma_{22}^2 + \left[\left(\frac{F_1^+ - F_1^-}{F_1^+F_1^-}\right)^2 + \left(\frac{F_2^+ - F_2^-}{F_2^+F_2^-}\right)^2 - \left(\frac{F_{12,45}^+ - F_{12,45}^-}{F_{12,45}^+F_{12,45}^-}\right)^2\right]\sigma_{11}\sigma_{22} + \frac{4\tau_{12}^2}{F_{12}^2}\right\}^{\frac{1}{2}} \le 1. \tag{6}$$

Criterion 5 (the Liu–Huang–Stout criterion ([32]))

$$\left\{F\sigma_{22}^2 + G\sigma_{11}^2 + H(\sigma_{11} - \sigma_{22})^2 + 2N\tau_{12}^2\right\}^{\frac{1}{2}} + \\ + I\sigma_{11} + J\sigma_{22} \le 1, \tag{7}$$

where

$$F = \frac{1}{2}\left\{\left(\frac{F_y^+ + F_y^-}{2F_y^+F_y^-}\right)^2 - \left(\frac{F_x^+ + F_x^-}{2F_x^+F_x^-}\right)^2\right\},$$

$$G = \frac{1}{2}\left\{\left(\frac{F_x^+ + F_x^-}{2F_x^+F_x^-}\right)^2 - \left(\frac{F_y^+ + F_y^-}{2F_y^+F_y^-}\right)^2\right\},$$

$$H = \frac{1}{2}\left\{\left(\frac{F_x^+ + F_x^-}{2F_x^+F_x^-}\right)^2 + \left(\frac{F_y^+ + F_y^-}{2F_y^+F_y^-}\right)^2\right\},$$

$$I = -\frac{F_x^+ - F_x^-}{2F_x^+F_x^-}, \quad J = -\frac{F_y^+ - F_y^-}{2F_y^+F_y^-}, \quad N = \frac{1}{2(F_{xy})^2}.$$

Criterion 6 (the Tsai–Wu criterion (Tsai, Wu, [35]))

$$\left(\frac{1}{F_x^+} + \frac{1}{F_x^-}\right)\sigma_{11} + \left(\frac{1}{F_y^+} + \frac{1}{F_y^-}\right)\sigma_{22} - \\ - \frac{1}{F_x^+F_x^-}\sigma_{11}^2 - \frac{1}{F_y^+F_y^-}\sigma_{22}^2 - \\ - \frac{1}{2}\sqrt{\frac{1}{F_x^+F_x^-}\frac{1}{F_y^+F_y^-}}\sigma_{11}\sigma_{22} - \frac{1}{F_{xy}^+F_{xy}^-}\tau_{12}^2 \le 1. \tag{8}$$

Criterion 7 (the Hoffman criterion [6])

$$\frac{1}{2}\left(\frac{1}{F_y^+F_y^-} - \frac{1}{F_x^+F_x^-}\right)\sigma_{11}^2 + \frac{1}{2}\left(\frac{1}{F_x^+F_x^-} - \frac{1}{F_y^+F_y^-}\right)\sigma_{22}^2 + \\ + \left(\frac{1}{F_x^+} + \frac{1}{F_x^-}\right)\sigma_{11} + \left(\frac{1}{F_y^+} + \frac{1}{F_y^-}\right)\sigma_{22} - \\ - \frac{1}{2}\left(\frac{1}{F_y^+F_y^-} + \frac{1}{F_x^+F_x^-}\right)(\sigma_{11} - \sigma_{22})^2 - \frac{1}{F_{xy}^+F_{xy}^-}\tau_{xy}^2 \le 1. \tag{9}$$

For criteria 2 and 3, the following condition is in place:

$$F_1 = \begin{cases} F_1^+ & \text{при } \sigma_{11} \ge 0, \\ F_1^- & \text{при } \sigma_{11} < 0, \end{cases} \quad F_2 = \begin{cases} F_2^+ & \text{при } \sigma_{22} \ge 0, \\ F_2^- & \text{при } \sigma_{22} < 0. \end{cases} \tag{10}$$

It should be noted that some sources do not put the minus sign with ultimate compressive stresses. To ensure consistency in notation with regard to the criteria and values calculated, assumed that compressive stres–ses (including ultimate compressive stresses) have the minus sign.

In defining the criteria, the following designations are accepted: $1, 2, 3$ – the orthogonal system of coordi–nates corresponding to the orthotropy axes of the mate–rial; $F_1^+, F_2^+$ – ultimate strength under tension in the $1, 2$ orthotropy directions [MPa]; $F_1^-, F_2^-$ – ultimate strength under compression in the $1, 2$ orthotropy direc–tions [MPa]; $F_{12}$ – ultimate strength in shear in the orthotropy plane [MPa]; $F_{12,45}^+, F_{12,45}^-$ – ultimate stren–gth in shear along the planes inclined at 45° to the principal directions [MPa]; $\sigma_{11}, \sigma_{22}, \sigma_{33}$ – normal stresses in the $1, 2, 3$ orthotropy directions [MPa]; $\tau_{12}, \tau_{13}, \tau_{23}$ – shear stresses in the $1O2, 1O3, 2O3$ planes [MPa].

If the 1, 2 orthotropy axes of the material do not coincide with the $x, y$ coordinate axes of the structure, then structural stresses $\sigma_x, \sigma_y, \tau_{xy}$ shall be adjusted to the 1, 2 orthotropy directions by using equations for coordinate system rotation. In such a case, the known values of the ultimate strength of the material can be used in strength criterion equations.

From now on, it will consider that the 1 and 2 orthotropy axes coincide with the $x, y$ axes of the accepted local coordinate system, respectively ($x, y, z$ – the orthogonal coordinate system in the middle surface of the shell structure; $x, y$ – the curvilinear coordinates oriented along the main shell curvature lines, $z$ – the coordinate oriented in the direction of the concavity, perpendicular to the middle surface).

## 2.2 Description of structures under consideration

T-10/UPE22-27 glass-fiber-reinforced plastic [2] as the structural material is selected. Its characteristics are given in Table 1.

**Table 1 T-10/UPE22-27 glass-fiber-reinforced plastic characteristics [2]**

| $E_1$, MPa | $\mu_{12}$ | $E_2$, MPa | $G_{12}$, MPa | $G_{13}$, MPa |
|---|---|---|---|---|
| $0.294 \cdot 10^5$ | 0.123 | $1.78 \cdot 10^4$ | $0.301 \cdot 10^4$ | $0.301 \cdot 10^4$ |
| $G_{23}$, MPa | $F_1^+$, MPa | $F_1^-$, MPa | $F_2^+$, MPa | $F_2^-$, MPa |
| $0.301 \cdot 10^4$ | 508 | –209 | 246 | –117 |

| $F_{12}$, MPa | $F^{+}_{12,45}$, MPa | $F^{-}_{12,45}$, MPa |
|---|---|---|
| 43 | 130 | –160 |

Table 2 shows the input parameters of a doubly-curved shallow shell under consideration.

**Table 2 Input parameters of a doubly-curved sallow shell**

| No. | $h$, m | $a$, m | $b$, m | $R_1$, m | $R_2$, m |
|---|---|---|---|---|---|
| 1 | 0.01 | 1.2 | 1.2 | 4.8 | 4.8 |

The orthotropic doubly-curved shell is simply supported along the contour (for $x = a_1$, $x = a$, $U = V = W = M_x = \Psi y = 0$; for $y = 0$, $y = b$, $U = V = W = M_x = \Psi y = 0$).

Computer simulation of the process of deformation of shell structures is important for many industries, including the construction and aircraft industry [36–39].

To unambiguously interpret these input parameters, Fig. 1 presents a general view of a doubly-curved shallow shell reinforced with stiffeners with the accepted local coordinate system.

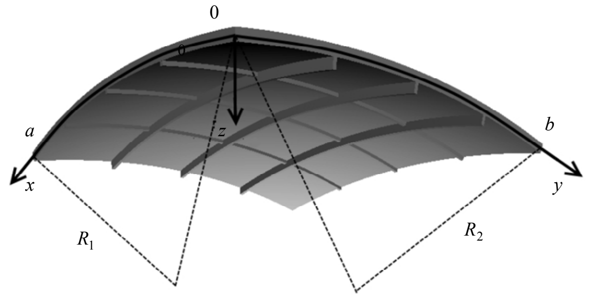


**Figure 1. Accepted local coordinate system**

The shells are square in plan, with pin support along the contour, and subjected to uniformly distributed tra–nsverse load $q$ directed along the normal to the surface.

The shells are reinforced with an orthogonal grid of stiffeners uniformly distributed throughout the structure. The width of the stiffeners is $r^j = r^i = 2h$, and the height is $h^j = h^i = 3h$. The distance between the stiffeners defined as $x_r$, with the outer stiffeners at a distance $0.5x_r$ from the edge of the structure.

### 2.3 Mathematical model and analysis algorithm

The functional of full potential deformation energy (the Lagrange functional) as the basis for the mathematical model of shell structure deformation is used.

In the case of static problems, the function can be represented as the difference between the potential energy of the system and the work of external forces:

$$E_s = E_s^0 + E_p^R = E_p^0 + E_p^R - \mathrm{A}, \tag{11}$$

where $E_s^0$ – the component of the statics functional, associated with the skin; $E_p^R$ – the potential energy of the system, associated with the stiffeners; $E_p^0$ – the potential energy of the system, associated with the skin; $\mathrm{A}$ – the work of external forces.

The part of the functional associated with the skin will be as follows:

$$E_s^0 = \frac{1}{2}\int_{a_1}^{a}\int_{0}^{b}\Big[N_x^0\varepsilon_x + N_y^0\varepsilon_y + \frac{1}{2}\left(N_{xy}^0 + N_{yx}^0\right)\gamma_{xy} + M_x^0\chi_1 + M_y^0\chi_2 + \left(M_{xy}^0 + M_{yx}^0\right)\chi_{12} + Q_x^0\left(\Psi_x - \theta_1\right) + Q_y^0\left(\Psi_y - \theta_2\right) - 2\left(P_x U + P_y V + qW\right)\Big]ABdxdy, \tag{12}$$

where $E_p^R$ depends on the method to arrange stiffeners, taking the following general form [40]:

$$E_p^R = \frac{1}{2}\int_{a_1}^{a}\int_{0}^{b}\Big[N_x^R\varepsilon_x + N_y^R\varepsilon_y + \frac{1}{2}\left(N_{xy}^R + N_{yx}^R\right)\gamma_{xy} + M_x^R\chi_1 + M_y^R\chi_2 + \left(M_{xy}^R + M_{yx}^R\right)\chi_{12} + Q_x^R\left(\Psi_x - \theta_1\right) + Q_y^R\left(\Psi_y - \theta_2\right)\Big]ABdxdy. \tag{13}$$

As a rule, if the external load is applied along the normal to the shell surface, then $P_x = P_{xsv}$, $P_y = P_{ysv}$ (dead load components), and their transverse component can be determined as follows:

$$q = q_0\left(a_{11} + a_{21}x + a_{31}x^2\right)\times \times\left(a_{12} + a_{22}y + a_{32}y^2\right) + q_{sv}, \tag{14}$$

where $q_0$ denotes the value of the applied load, MPa; $q_{sv}$ denotes the transverse shell dead load component, MPa.

This paper suggests using an algorithm based on the Ritz method and the solution continuation method with respect to the best parameter for studying shell struc–tures.

According to this algorithm, the Ritz method is applied to the function to reduce the variational problem to a system of nonlinear algebraic equations. For this purpose, the required functions are presented as follows:

$$\begin{aligned}
U &= U(x,y) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} U_{kl}X_1^kY_1^l,\\
V &= V(x,y) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} V_{kl}X_2^kY_2^l,\\
W &= W(x,y) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} W_{kl}X_3^kY_3^l,\\
\Psi_x &= \Psi_x(x,y) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} \Psi_{x\,kl}X_4^kY_4^l,\\
\Psi_y &= \Psi_y(x,y) = \sum_{k=1}^{\sqrt{N}}\sum_{l=1}^{\sqrt{N}} \Psi_{y\,kl}X_5^kY_5^l,
\end{aligned} \tag{15}$$

where $U_{kl} - \Psi_{y_{kl}}$ – unknown numerical parameters. Having applied functions (15) to functional (11), the derivatives with respect to unknown numerical para–meters $U_{kl} - \Psi_{y_{kl}}$ are founded. Thus, a system of nonlinear algebraic equations is derived.

To solve this system, the solution continuation method with respect to the best parameter is used. Verification of this algorithm is considered in detail in [41].

When the solution continuation method concerning the best parameter is used, the load/deflection curve is built step by step. At each step, the stress-strain state of the structure is analyzed. In the case of isotropic structures, it is sufficient to evaluate stress intensity, but in the case of orthotropic and anisotropic structures, it is required to apply special strength criteria. These criteria use constants of the ultimate values of stresses in the material. Besides, values of ultimate strength in different directions as well as values of ultimate strength under tension/compression and in shear, are different.

In the monograph [42], the authors analyzed stresses in different layers along the $z$ coordinate for shallow shells rectangular in plan. They showed that the maximum stresses occur on the outside of the shell at $z=-h/2$. That is why the shell reinforced with stiffeners on the convex side is more rigid than the shell reinforced with stiffeners on the concave side. However, for technological reasons, shells are often reinforced with stiffeners on the concavity as well.

To analyze strength criteria, stresses $\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz}$ at $z=-h/2$ are calculated.

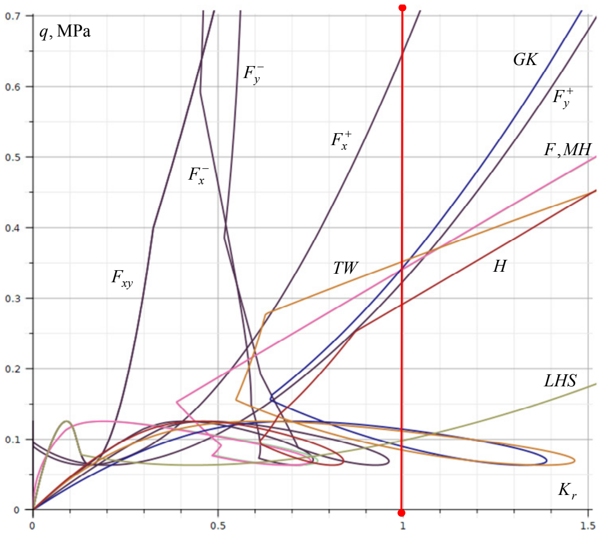


**Figure 2. Strength criterion curves for a doubly-curved shallow shell made of T-10/UPE22-27 glass-fiber-reinforced plastic with the 4 x 4 grid of stiffeners**

## 3. NUMERICAL RESULTS

### 3.1 Strength analysis of a glass-fiber-reinforced plastic shell

Next, the fulfillment of the strength conditions for a T-10/UPE22-27 glass-fiber-reinforced plastic shell with the $4\times4$ orthogonal grid of stiffeners by using several criteria. The author suggests building criterion/load curves and criteria fields under various loads as well as areas failing to meet the strength conditions in the post-buckling state. Fig. 2 shows a strength criterion vs. load diagram. The load corresponding to the point where a criterion curve exceeds 1 on the horizontal axis indicates the onset of failure to meet the strength conditions. In the diagram below, the maximum stress criterion is represented by six curves corresponding to its components. The sharp curve bends because the criterion's maximum value is calculated over the entire structure (i.e., those may be different points in different moments of loading). When one point with the maximum values changes to another, the curve changes direction since the criterion values in different points of the structure increase at different rates.

Table 3 shows the maximum stresses obtained based on various strength criteria. Among other things, such a spread in values is also due to the fact that, in terms of some criteria, maximum stresses are achieved before buckling and, in terms of other criteria – after buckling. In Fig. 2, this is represented by the loops.

**Table 3 Maximum stresses obtained based on various strength criteria**

| Strength criterion | | $q_{pr}$, MPa |
|---|---|---|
| Maximum stresses | $F_x^+$ | 0.6499 |
| | $F_x^-$ | – |
| | $F_y^+$ | 0.3249 |
| | $F_y^-$ | – |
| | $F_{xy}^+$ | 2.0468 |
| | $F_{xy}^-$ | 2.0468 |
| Mises–Hill | | 0.3425 |
| Fisher | | 0.3425 |
| Goldenblatt–Kopnov | | 0.1141 |
| Liu–Huang–Stout | | 0.0987 |
| Tsai–Wu | | 0.1113 |
| Hoffman | | 0.2942 |

Criterion fields under a load of 0.3883 MPa are shown in Fig. 3.

This load value was chosen because it exceeds the values of all ultimate loads $q_{pr}$ obtained according to various criteria, and allows us to analyze the areas of the structure where the strength conditions were violated. In addition, this value is close to the values of ultimate loads, which makes it possible to estimate exactly where the beginning of irreversible changes in the shell material occurred before these areas had time to expand.

According to the components of the criterion of maxi–mum stresses, the places of stress concentration are visible, which together can be seen in the images of other strength criteria. The highest values correspond to the sections at the edges of the structure (in the middle of the span).

All criteria show the absence of stress concentration in the center of the shell; also, the minimum stress values are observed at the corner points.

The Mises – Hill and Fisher criteria fields look al–most the same, corresponding to the minimum diffe–rence in their formulas (the difference lies in only one coefficient).

The nature of the Liu – Huang – Stout criterion field is similar to the Mises – Hill, and Fisher criteria but gives higher values and shows a smaller difference in values between the concentrations at the $x$ and $y$ edges.

The Tsai – Wu, and Hoffman criteria show stress concentrations at the edges of the structure to a lesser extent. It can be seen from them that the strength conditions are violated at the boundary of the structure along the $x$ axis, while the stresses at the boundary along the $y$ axis are practically insignificant.

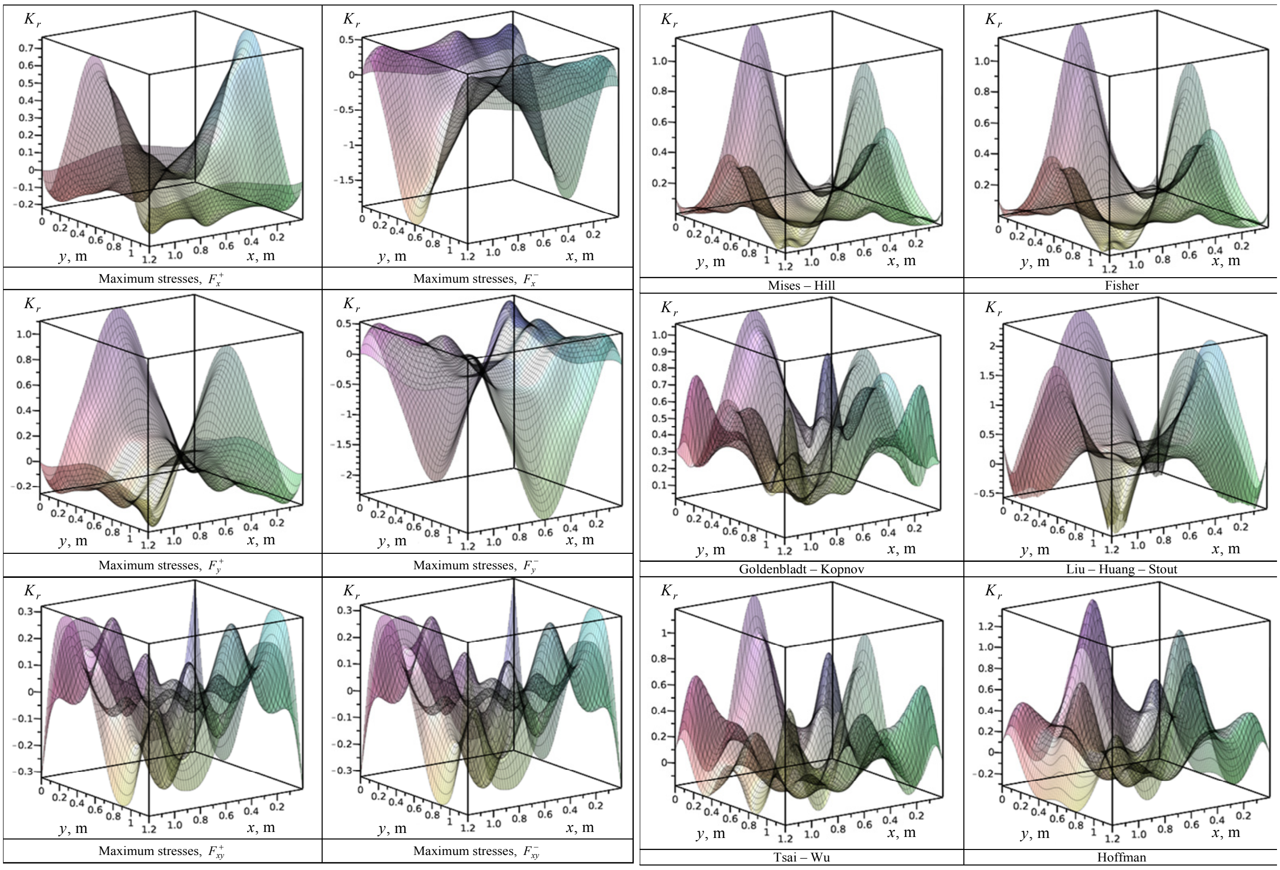


**Figure 3. Strength criterion fields under a load of 0.3883 MPa for a doubly-curved shallow shell**

The extension of areas failing to meet the strength conditions, based on some of the described criteria, at a load of 0.5869 MPa and a load of 1.056 MPa, are shown in Fig. 4. These load values were chosen due to the fact that it allows analyzing the process of increasing the areas of irreversible changes in the material: the first load value is taken slightly more than the founded maximum allowable load $q_{pr}$, and the second – before the strength conditions cease to be satisfied at all points of the structure.

The development of areas failing to meet the strength conditions indicate, first of all, the occurrence of damage to the material on the edge of the structure along the $x$ axis. This can be explained as follows: despite the structure under consideration being symmetrical in terms of geometry, reinforcement, and the applied load, orthotropic material is used. In the case of T-10/UPE22-27 glass-fiber-reinforced plastic, the ultimate strength along the 1 orthotropy axis (which coincides with the $x$ axis) is twice as high as that along direction 2. Thus, when tensile stresses occur along direction 2, areas failing to meet the strength conditions occur closer to the edges of the structure. As the load increases, areas are expanding and supplemented by similar areas near the other two edges. These areas develop similarly but with a "delay".

The results obtained based on different strength criteria are very similar. However, it should be noted that the Liu–Huang–Stout criterion comes into action earlier and is indicative of more significant changes in the material under the same loads. In terms of the Hoffman criterion, the areas expand more slowly and differ in shape. Besides, they even fail to develop along the 1 orthotropy axis under the loads under consideration.

The maximum stress criterion shall be considered as a set of all its components: in this case, only $F_y^+$ and $F_x^+$ are significant, and the cumulative area failing to meet the strength conditions can be obtained by their combination.

## 4. CONCLUSION

The paper described a method of strength analysis with regard to the materials of thin-walled shell structures reinforced with stiffeners, using doubly-curved shallow shells as an example. The applicability of the following seven strength criteria was analyzed: the maximum stress criterion, the Mises–Hill criterion, the Fisher criterion, the Goldenblatt–Kopnov criterion, the Liu–Huang–Stout criterion, the Tsai–Wu criterion, and the Hoffman criterion. Three-dimensional graphs for strength criteria under a given load are also presented. Besides, the development of areas failing to meet the strength conditions by different criteria is analyzed. The suggested presentation format with regard to data on areas failing to meet the strength conditions and a strength criterion diagram is new and makes it possible to evaluate the state of the structure visually.

The sequence of actions indicated in work and, especially, the format for presenting data on the state of the structure (fields of strength criteria, graphs of dependence of strength criteria on load, graphs of the development of areas of non-fulfillment of strength conditions) are new and represent a scientific novelty.

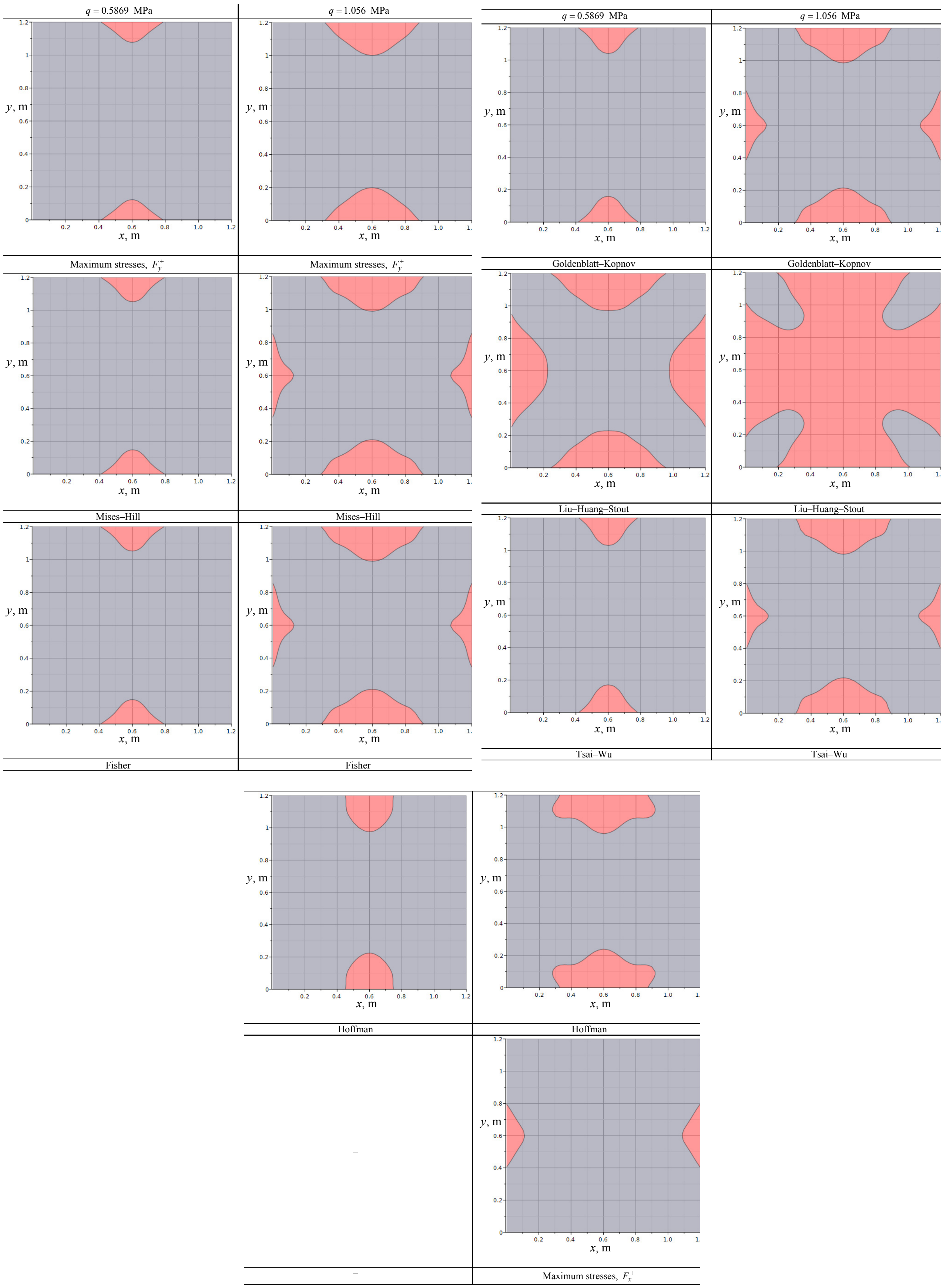


**Figure 4. Development of areas failing to meet the strength conditions at a load of 0.5869 MPa and a load of 1.056 MPa for a doubly-curved shallow shel**

The practical significance of the work lies in the fact that the proposed method and the developed computa–tional computer program for studying the stress-strain state, strength, and buckling of shells stiffened with ribs made of orthotropic materials under static loading can be used in design organizations and further scientific research.

## REFERENCES


[1] Alankaya, V., Oktem, A.S.: Static analysis of laminated and sandwich composite doubly-curved shallow shells, Steel and Composite Structures, Vol. 20, No. 5, pp. 1043–1069, 2016. DOI: 10.12989/SCS.2016.20.5.1043.

[2] Tyshkevich, V.N.: Choice of strength criterion for pipes made of reinforced plastics, Izvestiya VolgGTU, No. 5, pp. 76–79, 2011.

[3] Smerdov, A.A.: Rational Choice of Angle-Ply Composites for Identification of the Elastic Characteristics of Unidirectional Composites, Mechanics of Composite Materials, Vol. 54, No. 3, pp. 269–280, 2018. DOI: 10.1007/s11029-018-9738-4.

[4] Sengupta, J., Ghosh, A., Chakravorty, D.: Progressive Failure Analysis of Laminated Composite Cylindrical Shell Roofs, Journal of Failure Analysis and Prevention, Vol. 15, No. 3, pp. 390–400, 2015. DOI: 10.1007/s11668-015-9951-6.

[5] Niu, J., Liu, G., Tian, J., Zhang, Y.-X., Meng, L.-P.: Comparison of yield strength theories with experimental results, Engineering Mechanics, Vol. 31, No. 1, pp. 181–187, 2014. DOI: 10.6052/j.issn.1000-4750.2012.09.0622.

[6] Abrosimov, N.A., Elesin, A.V.: Numerical analysis of dynamic strength of composite cylindrical shells under multiple-pulse exposures, PNRPU Mechanics Bulletin, No. 4, pp. 7–19, 2016. DOI: 10.15593/perm.mech/2016.4.01.

[7] Grinevich, D.V., Yakovlev, N.O., Slavin, A.V.: The Criteria of the Failure of Polymer Matrix Composites (review), Proceedings of VIAM, No. 7, pp. 92–111, 2019. DOI: 10.18577/2307-6046-2019-0-7-92-111.

[8] Oreshko, E.I., Erasov, V.S., Grinevich, D.V., Shershak, P.V.: Review of Criteria of Durability of Materials, Proceedings of VIAM, No. 9, pp. 108–126, 2019. DOI: 10.18577/2307-6046-2019-0-9-108-126.

[9] Yokoyama, N.O., Donadon, M.V., de Almeida, S.F.M.: A numerical study on the impact resistance of composite shells using an energy based failure model, Composite Structures, Vol. 93, No. 1, pp. 142–152, 2010. DOI: 10.1016/j.compstruct.2010.06.006.

[10] Komarov, V.A., Kishov, E.A., Charkviani, R.V., Pavlov, A.A.: Numerical and Experimental Study of the Strength of Fabric Carbon-Epoxy Composite Structures, VESTNIK of Samara University. Aerospace and Mechanical Engineering, Vol., 14, No. 2, pp. 106–112, 2015. DOI: 10.18287/2412-7329-2015-14-2-106-112.

[11] Polilov, A.N., Tatus, N.A.: Experimental Substantiation of Strength Criteria for FRP Showing Directional Type of Fracture, PNRPU Mechanics Bulletin, No. 2, pp. 140–166, 2012.

[12] Garnich, M.R., Akula, V.M.K.: Review of Degradation Models for Progressive Failure Analysis of Fiber Reinforced Polymer Composites, Applied Mechanics Reviews, Vol. 62, No. 1, pp. 010801, 2009. DOI: 10.1115/1.3013822.

[13] Günel, M., Kayran, A.: Nonlinear progressive failure analysis of open-hole composite laminates under combined loading, Journal of Sandwich Structures & Materials, Vol. 15, No. 3, pp. 309–339, 2013. DOI: 10.1177/1099636213483651.

[14] Nekliudova, E.A., Semenov, A.S., Melnikov, B.E., Semenov, S.G.: Experimental research and finite element analysis of elastic and strength properties of fiberglass composite material, Magazine of Civil Engineering, No. 3, pp. 25–39, 2014. DOI: 10.5862/MCE.47.3.

[15] Paczos, P., Zielnica, J.: Critical load of a bilayered orthotropic elastic–plastic conical shell with the change of the shell basic surface location, Thin-Walled Structures, Vol. 45, No. 10-11, pp. 911–915, 2007. DOI: 10.1016/j.tws.2007.08.032.

[16] Maksimović, K., Maksimović, M., Vasović-Maksimović, I., Rašuo, B., Maksimović, S.: Postbuckling and failure analysis of layered composite panels, FME Transactions, Vol. 48, No. 2, pp. 447–453, 2020. DOI: 10.5937/fme 2002447M.

[17] Galicki, J., Czech, M.: A new approach to formulate the general strength theories for anisotropic discontinuous materials. Part A: The experimental base for a new approach to formulate the general strength theories for anisotropic materials on the basis of wood, Applied Mathematical Modelling, Vol. 37, No. 3, pp. 815–827, 2013. DOI: 10.1016/j.apm.2012.03.004.

[18] Pietropaoli, E.: Progressive Failure Analysis of Composite Structures Using a Constitutive Material Model (USERMAT) Developed and Implemented in ANSYS ©, Applied Composite Materials, Vol. 19, No. 3-4, pp. 657–668, 2012. DOI: 10.1007/s10443-011-9220-0.

[19] Shokrieh, M.M., Karamnejad, A.: Investigation of Strain Rate Effects on the Dynamic Response of a Glass/Epoxy Composite Plate Under Blast Loading by Using the Finite-Difference Method, Mechanics of Composite Materials, Vol. 50, No. 3, pp. 295–310, 2014. DOI: 10.1007/s11029-014-9415-1.

[20] Shroff, S., Kassapoglou, C.: Progressive failure modelling of impacted composite panels under compression, Journal of Reinforced Plastics and Composites, Vol. 34, No. 19, pp. 1603–1614, 2015. DOI: 10.1177/0731684415592485.

[21] Baryshev, A.N., Tsvetkov, S.V.: Building the Strength Criterion of Carbon-Carbon Composite Material of the 4DL Type Under Triaxial Stress, BMSTU Journal of Mechanical Engineering, No. 6, pp. 78–85, 2017. DOI: 10.18698/0536-1044-2017-6-78-85.

[22] Kolupaev, V.A., Yu, M.-H., Altenbach, H.: Visualization of the Unified Strength Theory, Archive of Applied Mechanics, Vol. 83, No. 7, pp. 1061-1085, 2013. DOI: 10.1007/s00419-013-0735-8

[23] Tsvetkov, S.V., Kulish, G.G.: Strength criteria for unidirectional organoplastic under triaxial stress, Herald of the Bauman Moscow State Technical University. Series Mechanical Engineering, No. SP, pp. 19–28, 2011.

[24] Yu, M.-H.: Advances in strength theories for materials under complex stress state in the 20th Century, Applied Mechanics Reviews, Vol. 55, No. 3, pp. 169–218, 2002. DOI: 10.1115/1.1472455.

[25] Korsun, V.I., Karpenko, S.N., Makarenko, S.Yu., Nedorezov, A.V.: Modern Strength Criteria for Concrete under Triaxial Stress States, Construction and Reconstruction, No. 5, pp. 16–30, 2021. DOI: 10.33979/2073-7416-2021-97-5-16-30.

[26] Bleyer, J., de Buhan, P.: A numerical approach to the yield strength of shell structures, European Journal of Mechanics – A/Solids, Vol. 59, pp. 178–194, 2016. DOI: 10.1016/j.euromechsol.2016.03.002.

[27] Kalnins, A., Updike, D.P.: Limit Pressures of Cylindrical and Spherical Shells, Journal of Pressure Vessel Technology, Vol. 123, No. 3, pp. 288–292, 2001. DOI: 10.1115/1.1367273.

[28] Mellor, P.B.: The ultimate strength of thin-walled shells and circular diaphragms subjected to hydrostatic pressure, International Journal of Mechanical Sciences, Vol. 1, No. 2-3, pp. 216–228, 1960. DOI: 10.1016/0020-7403(60)90041-2.

[29] Noh, H.C.: Ultimate strength of large scale reinforced concrete thin shell structures, Thin-Walled Structures, Vol. 43, No. 9, pp. 1418–1443, 2005. DOI: 10.1016/j.tws.2005.04.004

[30] Sun, H.-H., Tan, P.-L.: Background of ABS Buckling Strength Assessment Criteria for Cylindrical Shells in Offshore Structures, Journal of Offshore Mechanics and Arctic Engineering, Vol. 130, No. 2, pp. 021012, 2008. DOI: 10.1115/1.2913349.

[31] Yan, L., Junhai, Z., Ergang, X., Xueye, C.: Research on burst pressure for thin-walled elbow and spherical shell made of strength differential materials, Materials Research Innovations, Vol. 19, No. 5, pp. 80–87, 2015. DOI: 10.1179/1432891715Z.0000000001340.

[32] Liu, C., Huang, Y., Stout, M.G.: On the asymmetric yield surface of plastically orthotropic materials: A phenomenological study, Acta Materialia, Vol. 45, No. 6, pp. 2397–2406, 1997. DOI: 10.1016/S1359-6454(96)00349-7.

[33] Fisher, L.: How to predict structural behavior of R.P. Laminates, Modern Plastics, Vol. 37, No. 6, pp. 122–209, 1960.

[34] Goldenblat, I.I., Kopnov, V.A.: Strength criterion for anisotropic materials, Proceedings of the Academy of Sciences of the USSR. Mechanics, No. 6, pp. 77–83, 1965.

[35] Tsai, S.W., Wu, E.M.: A General Theory of Strength for Anisotropic Materials, Journal of Composite Materials, Vol. 5, No. 1, pp. 58–80, 1971. DOI: 10.1177/002199837100500106.

[36] Bakshi, K. et al.: Relative static and dynamic per–formances of composite conoidal shell roofs, Steel and Composite Structures, Vol. 15, No. 4, pp. 379–397, 2013. DOI: 10.12989/SCS.2013. 15.4.379.

[37] Darılmaz, K.: Static and free vibration behaviour of orthotropic elliptic paraboloid shells, Steel and Composite Structures, Vol. 23, No. 6, pp. 737–746, 2017. DOI: 10.12989/SCS.2017.23.6.737.

[38] Dinulović, M., Rašuo, B., Slavković, N., Karić, Đ.: Analysis of aspect and taper ratio on aeroelastic stability of composite shells, FME Transactions, Vol. 50, No. 4, pp. 732–744, 2022. DOI: 10.5937/fme2204732D.

[39] Stankovic, M., Ristic, M., Simonovic, A., Jovanovic, M.: Buckling behaviour of dented aluminium alloy cylindrical shell subjected to uniform axial compression, FME Transaction, Vol. 45, No. 3, pp. 441–447, 2017. DOI: 10.5937/fmet1703441S.

[40] Semenov, A.: Strength of Steel Shell Cylindrical Panels Reinforced with an Orthogonal Grid of Stiffeners, Journal of Applied and Computational Mechanics, Vol. 8, No. 2, pp. 723–732, 2022. DOI: 10.22055/jacm.2022.38968.3317.

[41] Semenov, A.A., Leonov, S.S.: The continuous method of solution continuation with respect to the best parameter in the calculation of shell structures, Uchenye Zapiski Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki, Vol. 161, No. 2, pp. 230–249, 2019. DOI: 10.26907/2541-7746.2019.2.230-249.

[42] Karpov, V.V., Ignatiev, O.V., Salnikov, A.Yu.: Nonlinear mathematical models of deformation of shells of variable thickness and algorithms for their study. SPbGASU, SPb, 2002.


**NOMENCLATURE**

| | |
|---|---|
| $A, B$ | Lame parameters describing the shell geometry |
| $E_1, E_2$ | elastic moduli |
| $E_s$ | functional of full potential defor–mation energy of the shell structure |
| $E_s^0$ | functional of full potential deformation energy of the skin |
| $E_p^R$ | potential deformation energy of the stiffeners |
| $f(z)$ | function describing the distribution of stresses $\tau_{xz}$ and $\tau_{yz}$ through the shell thickness |
| $F_1^+, F_2^+$ | ultimate strength under tension in the 1, 2 orthotropy directions |
| $F_1^-, F_2^-$ | ultimate strength under compres–sion in the 1, 2 orthotropy directions |
| $F_{12}$ | ultimate strength in shear in the orthotropy plane |
| $F_{12,45}^+, F_{12,45}^-$ | ultimate strength in shear along the planes inclined at 45° to the principal directions |
| $G_{12}, G_{13}, G_{23}$ | shear modules |

| | |
|---|---|
| $h^i, h^j$ | the height of the stiffeners |
| $k_x, k_y$ | main curvatures of the shell along the $x$ $y$ axes |
| $K_r$ | yield strength criterion |
| $M_x^0, M_y^0, M_{xy}^0, M_{yx}^0$ | moments occurring in the skin |
| $M_x^R, M_y^R, M_{xy}^R, M_{yx}^R$ | forces and moments occurring in the stiffeners |
| $N_x^0, N_y^0, N_{xy}^0, N_{yx}^0$ | forces occurring in the skin |
| $N_x^R, N_y^R, N_{xy}^R, N_{yx}^R$ | forces and moments occurring in the stiffeners |
| $N$ | number of terms in the expansion of the Ritz method |
| $q, P_x, P_y$ | load components |
| $q_0$ | the value of the applied load |
| $q_{sv}$ | transverse shell dead load component |
| $q_{pr}$ | maximum permissible load (strength); |
| $Q_x^0, Q_y^0$ | transverse forces in the planes $x\mathrm{O}z$ and $y\mathrm{O}z$, occurring in the skin |
| $Q_x^R, Q_y^R$ | transverse forces in the planes $x\mathrm{O}z$ and $y\mathrm{O}z$, occurring in the stiffeners |
| $r^i, r^j$ | width of the stiffeners |
| $U, V, W$ | displacement functions |
| $U_{kl} - \Psi_{y\,kl}$ | unknown numerical parameters |
| $x_r$ | the distance between the stiffeners |
| A | the work of external forces |
| $\gamma_{xy}$ | shear deformation in the $x\mathrm{O}y$ plane |
| $\chi_1, \chi_2, \chi_{12}$ | functions of curvature and torsional change |
| $\Psi_x, \Psi_y$ | functions of the normal rotation angles in the $x\mathrm{O}z$ and $y\mathrm{O}z$ planes, respectively |
| $\varepsilon_x . \varepsilon_y$ | deformations of elongation along the $x$, $y$ coordinates of the middle surface |
| $\mu_{12}, \mu_{21}$ | Poisson's ratios |
| $\tau_{12}, \tau_{13}, \tau_{23}$ | shear stresses in the 1O2, 1O3, 2O3 planes |
| $\tau_{xy}, \tau_{xz}, \tau_{yz}$ | shear stresses in the plane $x\mathrm{O}y$, $x\mathrm{O}z$ and $y\mathrm{O}z$ |
| $\sigma_{11}, \sigma_{22}, \sigma_{33}$ | normal stresses in the 1, 2, 3 orthotropy directions |
| $\sigma_x, \sigma_y$ | the normal stresses in the directions of axes $x$, $y$ |

## МЕТОДА АНАЛИЗЕ ЧВРСТОЋЕ ЗА ДВОСТРУКО ЗАКРИВЉЕНЕ УКРУЋЕНЕ ОРТОТРОПНЕ ШКОЉКЕ ПРЕМА РАЗЛИЧИТИМ ТЕОРИЈАМА ЧВРСТОЋЕ

**А.А. Семенов**

У раду је предложена метода анализе чврстоће материјала танкозидних шкољкастих конструкција ојачаних елементима за укрућење. Шкољке које се разматрају биле су од ортотропних материјала. Аутори анализирају применљивост следећих седам критеријума чврстоће: критеријум максималног напрезања, критеријум Мизес–Хил, критеријум Фишер, критеријум Голденблат–Копнов, критеријум Лиу–Хуанг–Стаут, критеријум Цаи–Ву и Хофманов критеријум. критеријум. Током истраживања раз–матране су двоструко закривљене плитке шкољке квадратне у плану. Коришћен је геометријски нелинеарни математички модел деформације шкољке, који разматра попречне смицање. Прора–чуни су засновани на карактеристикама Т-10/УПЕ22-27 пластике ојачане стакленим влакнима. Метода се ослања на израчунавање вредности неколико критеријума чврстоће у сваком кораку конструкцијског оптерећења и анализу развоја подручја која не испуњавају услове чврстоће како се оптерећење повећава.