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Adaptive One-Step-Ahead Iterative Learning Control Based on Generalized Forgetting Recursive Least Squares

This paper considers adaptive one-step-ahead Iterative Learning Control (ILC) for systems described by a Deterministic Autoregressive Moving Average (DARMA) model. The unknown model parameters are estimated using the recently introduced generalized forgetting recursive least squares method. The system is assumed to be minimum phase. The paper relaxes two fundamental ILC assumptions: identical initial conditions and identical reference trajectories. Instead, the initial regressor vector is considered random, and the reference trajectory varies across iterations. The proposed algorithm has a 2D structure, where parameter updates are performed iteratively along the iteration axis and recursively along the time axis.

Keywords: adaptive control, iterative learning control, random initial condition, iteration-varying reference trajectories, stability of adaptive systems.

1. INTRODUCTION

Iterative Learning Control (ILC) is designed to improve the control performance of repetitive processes by utilizing past control information [1,2]. Specifically, ILC computes the current control input based on the previous control input and tracking error, allowing the system to learn from experience and enhance tracking performance over iterations.

The seminal work by Arimoto et al. marked the beginning of extensive research in this field [3], leading to numerous applications [4-6]. Recent studies have further demonstrated the theoretical and practical relevance of ILC in robotics and feedback-controlled systems [7-9].

In particular, standard and norm-optimal ILC strategies have been compared in terms of tracking performance and robustness [10], while ILC has also been experimentally implemented in a walking-beam motion system with a flexible coupled robotic mechanism [11]. Additionally, the implementation of ILC in feedback-controlled systems has been discussed as a practically important framework in which the learning signal is incorporated into an existing feedback control structure to improve tracking performance [12].

Unlike conventional feedback control, which improves performance along the time axis, ILC enhances control precision along the iteration axis. It can also be combined with feedback control to achieve better overall performance. In such configurations, the feedback controller serves as an inner loop for system

stability, while ILC functions as an outer loop to refine tracking accuracy [12,13]. Mathematically, ILC exhibits characteristics of a 2D system [14,15], making 2D system theory a suitable framework for ILC synthesis.

Standard ILC operates under the assumption of perfectly repeating conditions, including a fixed time interval, identical initial states, identical reference trajectories, and a consistent operating environment. However, some works have addressed relaxation of these constraints. For example, [16] investigates ILC for discrete-time linear systems where the reference trajectory length varies randomly across iterations [17], thereby removing the requirement for identical reference trajectories. A detailed analysis of ILC design under relaxed conditions is provided in [13], covering cases where: (i) strict identical initial conditions are not required, (ii) the reference trajectory varies from iteration to iteration, and (iii) the reference trajectory length remains constant.

A promising approach for ILC design is adaptive control for discrete-time systems [18]. References [13,19] introduce a new discrete-time adaptive ILC framework for nonlinear systems. Additionally, [20] presents an adaptive ILC scheme for nonlinear, non-affine discrete systems, addressing scenarios where the reference trajectory length and initial conditions vary randomly, and system uncertainties are present.

In this paper, we develop an adaptive one-step-ahead ILC scheme based on the generalized forgetting recursive least squares method [21]. A key concept in system identification is the criterion (cost) function [22]. Several extensions of standard least squares methods exist, such as variable forgetting factors [23], directional forgetting [24,25], and exponential resetting [26]. A generalized criterion function that encompasses these modifications has been explored, with initial developments in continuous-time least squares presented in [27].

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The extension of this approach to discrete-time MIMO systems is discussed in [21], where the criterion function includes three components: a loss term, a forgetting term, and a regularization term. Furthermore, [28] demonstrates that an extended generalized forgetting recursive least squares framework can lead to an adaptive Kalman filter.

This paper presents the design of an adaptive ILC scheme for discrete-time DARMA (Deterministic Auto-regressive Moving Average) systems with constant parameters. The unknown parameters are estimated using the generalized forgetting least squares method. For the case where the reference trajectory length is constant, strict identical initial conditions are not required, and the reference trajectory varies across iterations.

The proposed method has an indirect adaptive-control structure embedded within the ILC framework. Input-output measurements are processed by the generalized-forgetting RLS algorithm to update the estimates of the unknown DARMA parameters. These updated parameter estimates are then used directly in the one-step-ahead control law. The resulting control input produces new plant data, which are subsequently used in the next identification and learning update. Consequently, system identification and control are not treated as two independent procedures but form a coupled two-dimensional adaptive learning mechanism: adaptation proceeds recursively along the time axis and iteratively along the learning-iteration axis.

The novelty of the present work lies primarily in its integration into an adaptive one-step-ahead iterative learning control framework for discrete-time DARMA systems under relaxed trial-to-trial repeatability assumptions, where key contributions are:

- (i) integration of generalized-forgetting recursive least-squares identification with one-step-ahead iterative learning control;
- (ii) development of a two-dimensional learning structure, in which parameter adaptation proceeds recursively along the time axis and iteratively along the learning-iteration axis;
- (iii) accommodation of bounded, non-identical initial regressor vectors and iteration-varying reference trajectories of fixed trial length;
- (iv) proof
 - of uniform boundedness of the parameter estimates;
 - of convergence of the tracking error to zero for every fixed time instant;
 - of boundedness of the plant input and output.

The proposed algorithm is also numerically investigated for a DARMA system with time-varying parameters. However, the convergence results established in this paper apply to the constant-parameter case; a rigorous extension to time-varying parameters would require additional assumptions on the parameter variation and a separate convergence analysis.

In general, this paper considers a special form of the algorithm proposed in [21] for the identification of time-varying systems. The proposed adaptive iterative learning control algorithm, together with its stability analysis, represents a novel contribution. Moreover, to the best of the authors' knowledge, systems described by the DARMA model have not been addressed in the existing literature.

2. GENERALIZED FORGETTING LEAST SQUARES

In standard least squares (LS) [22], the eigenvalues of the covariance matrix decrease monotonically over time, which may result in arbitrarily small values and lead to sluggish adaptation. Furthermore, standard LS cannot track time-varying parameters [26]. Various extensions have been developed to eliminate the monotonicity of the covariance matrix [23-26]. All these extensions fall under the category of general forgetting algorithms [21, 27], developed for MIMO systems. It is of interest to derive a criterion function that leads to such a general forgetting algorithm.

Least squares estimation can be viewed as a Gaussian process and interpreted as a machine learning algorithm [29,30].

In this context, the forgetting factor can be associated with the kernel function in kernel learning methods. We consider the standard regression model:

$$y(t) = \varphi^T(t) \theta(t), \varphi(t), \theta(t) \in \mathbb{R}^l, y(t) \in \mathbb{R} \quad (1)$$

where $y(t)$ is the measured scalar output, $\varphi(t)$ is the information vector (vector of input-output measurements), θ is the unknown parameter vector, and t represents discrete time. In (1), we would like to emphasize the more general case of a time-varying system ($\theta = \theta(t)$).

The generalized identification criterion for SISO systems is given by:

$$J_N(\theta) = \sum_{t=1}^N \alpha(t) [y(t) - \varphi^T(t)]^2 - \sum_{t=1}^N [\theta(t) - \theta]^T F(t) [\theta(t) - \theta] + [\theta - \hat{\theta}(0)]^T P^{-1}(0) [\theta - \hat{\theta}(0)] \quad (2)$$

The first term in (2) represents the prediction error weighted by $\alpha(t)$. The inclusion of $\alpha(t)$ in the criterion (2) allows different weights to be assigned to different observations. The second term is a forgetting term, where the matrix $F(t)$ is positive semidefinite and belongs to, $\mathbb{R}^{l \times l}$, $F(t) \in \mathbb{R}^{l \times l}$.

The third term in (2) is a regularization term. Since system identification is an inverse problem, it may suffer from numerical instability. Regularization is introduced to address this issue [31,32], where the matrix $P(0)$ is positive definite.

A recursive algorithm can be derived by minimizing the functional (2):

$$\frac{dJ_N(\theta)}{d\theta} = 0 \quad (3)$$

Following a similar approach as in [28], we obtain:

$$P^{-1}(t) = P^{-1}(t-1) - F(t-1) + \alpha(t) \varphi(t) \varphi^T(t) \quad (4)$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + \alpha(t) P(t) \varphi(t) (y(t) - \hat{\theta}^T(t-1) \varphi(t)) \quad (5)$$

where $\hat{\theta}(t)$ denotes the estimate of the unknown parameter vector θ at discrete time t , $P(t) \left(P(t) \in \mathbb{R}^{l \times l} \right)$ is

the symmetric positive-definite covariance matrix, and $P(0)$ denotes its prescribed symmetric positive-definite initial value, which also appears as the regularization matrix in criterion (2).

Remark 1. The condition that guarantees a unique global minimum for $J_M(\theta)$ is given in [28]:

$$F(t) < P^{-1}(0) + \sum_{i=1}^t \left(-F(i-1) + \alpha(i) \varphi(i) \varphi^T(i) \right) \quad (6)$$

Remark 2. Setting $F(t-1) = 0_{l \times l}$ and $\alpha(t) = 1$ in (4) and (5) leads to:

$$P^{-1}(t) = P^{-1}(t-1) + \varphi(t) \varphi^T(t) \quad (7)$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t) \varphi(t) \left(y(t) - \hat{\theta}^T(t-1) \varphi(t) \right) \quad (8)$$

which corresponds to the standard recursive least squares algorithm.

For practical applications, it is preferable to directly update $P(t)$ instead of using (4). This is achieved through the following lemma.

Lemma 1. (Matrix Inversion Lemma) ([22]; p. 19). Let A , B , C and D be matrices of compatible dimensions, with A and C nonsingular and with the indicated inverses well defined. Then:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B \left(DA^{-1}B + C^{-1} \right)^{-1} DA^{-1}.$$

Lemma 2 For all $t \geq 0, \theta \in \mathbb{R}^l, \varphi(i) \in \mathbb{R}^l$, and $u(t)$,

$y(t) \in \mathbb{R}$, we assume that :

1° $\alpha(t) > 0$

2° The matrix $P(t) \in \mathbb{R}^{l \times l}$ is symmetric positive definite.

3° The matrix $F(t) \in \mathbb{R}^{l \times l}$ is symmetric, positive semidefinite, and satisfies:

$$F(t-1) < P^{-1}(0) + \sum_{i=1}^{t-1} \left(-F(i-1) + \alpha(i) \varphi(i) \varphi^T(i) \right)$$

Under the assumptions 1°–3° of Lemma 2, from relation (4) it follows that

$$A) \ U(t-1) = \left(P^{-1}(t-1) F(t-1) \right)^{-1} > 0, \quad \forall t \geq 1, 2, 3, \dots$$

$$P(t) = U(t-1) - \frac{U(t-1) \varphi(t) \varphi^T(t) U(t-1)}{\alpha^{-1}(t) + \varphi^T(t) U(t-1) \varphi(t)}$$

Proof: From (4), we obtain:

$$P^{-1}(1) = P^{-1}(0) - F(0) + \alpha(1) \varphi(1) \varphi^T(1) \quad (9)$$

$$\begin{aligned} P^{-1}(2) &= P^{-1}(1) - F(1) + \alpha(2) \varphi(2) \varphi^T(2) = \\ &= P^{-1}(0) - (F(0) + F(1)) + \alpha(1) \varphi(1) \varphi^T(1) + \\ &\quad + \alpha(2) \varphi(2) \varphi^T(2) \\ &\quad \vdots \end{aligned}$$

$$P^{-1}(t-1) = P^{-1}(0) + \sum_{i=1}^{t-1} \left(-F(i-1) + \alpha(i) \varphi(i) \varphi^T(i) \right)$$

Using this relation and assumption (iii) of Lemma 2, we conclude:

$$P^{-1}(t-1) - F(t-1) > 0, \quad \forall t = 1, 2, 3, \dots \quad (10)$$

From this, we deduce:

$$U(t-1) = \left(P^{-1}(t-1) - F(t-1) \right)^{-1} > 0, \quad \forall t = 1, 2, 3, \dots \quad (11)$$

This proves part A) of the lemma. Now, following Lemma 1, we introduce the following:

$$\begin{aligned} A &= P^{-1}(t-1) - F(t-1) \quad B = \varphi(t) \\ C &= \alpha(t), \quad D = \varphi^T(t) \end{aligned} \quad (12)$$

From relation (4) and Lemma 1, we get:

$$\begin{aligned} P(t) &= \left[P^{-1}(t-1) - F(t-1) + \alpha(t) \varphi(t) \varphi^T(t) \right]^{-1} = \\ &= \left(P^{-1}(t-1) - F(t-1) \right)^{-1} - \left(P^{-1}(t-1) - F(t-1) \right)^{-1} \varphi(t) \cdot \\ &\quad \cdot \left(\alpha^{-1}(t) + \varphi^T(t) \left(P^{-1}(t-1) - F(t-1) \right)^{-1} \varphi(t) \right)^{-1} \cdot \\ &\quad \cdot \left(P^{-1}(t-1) - F(t-1) \right)^{-1} \end{aligned} \quad (13)$$

finally, combining relations (11) and (13), we obtain:

$$P(t) = U(t-1) - \frac{U(t-1) \varphi(t) \varphi^T(t) U(t-1)}{\alpha^{-1}(t) + \varphi^T(t) U(t-1) \varphi(t)} \quad (14)$$

Thus, the lemma is proved.

3. DISCRETE-TIME ADAPTIVE ILC

We assume the system is described by the DARMA model with constant parameters. Usually, differential operator calculus is used for manipulating linear differential equations with constant coefficients. Here, (q) operator denotes analog operator for difference equation. For the k -th iteration learning, we have

$$A(q^{-1}) y_k(t) = q^{-1} B(q^{-1}) u_k(t) \quad (15)$$

where $t \in [0, 1, \dots, T]$ denotes discrete time within one trial whereas $k = 1, 2, \dots$ represents the learning-iteration number, and $q^{-1}x(t) = x(t-1)$ is the backward-shift operator along the discrete-time axis.

The polynomials $A(q^{-1})$ and $B(q^{-1})$ are given by

$$A(q^{-1}) = 1 + a_1 q^{-1} + \dots + a_n q^{-n} \quad (16)$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \dots + b_m q^{-m} \quad (17)$$

From relation (15), we obtain

$$\begin{aligned} y_k(t+1) &= -a_1 y_k(t) - \dots - a_n y_k(t-n+1) + \\ &\quad + b_1 u_k(t-1) + \dots + b_m u_k(t-m) + b_0 u_k(t) \end{aligned} \quad (18)$$

This can be rewritten as

$$y_k(t+1) = \left(\varphi_k^0(t) \right)^T \theta^0 + b_0 u_k(t) \quad (19)$$

where

$$\varphi_k^0(t) = \begin{bmatrix} -y_k(t), -y_k(t-1), \dots, \\ -y_k(t-n+1), u_k(t-1), \dots, u_k(t-m) \end{bmatrix}^T$$

$$\theta^0 = [a_1, \dots, a_n, b_1, \dots, b_m]^T$$

The one-step ahead control for SISO systems is obtained by minimizing the following functional [18]:

$$J_{\theta,k}(t+1) = (y_k(t+1) - r_{d,k}(t+1))^2 \quad (20)$$

where $r_{d,k}(\cdot)$ is the reference trajectory for the k^{th} iteration. Optimal control follows from

$$\frac{\partial J_{\theta,k}(t+1)}{\partial u_k(t)} = 0 \quad (21)$$

From relation (21), we get

$$r_{d,k}(t+1) = y_k(t+1) \quad (22)$$

Using relations (19) and (22), the optimal controller for the k^{th} iteration is

$$u_k(t) = \frac{1}{b_0} (r_{d,k}(t+1) - (\varphi_k^0(t))^T \theta^0) \quad (23)$$

Let us define

$$\theta = \begin{bmatrix} (\theta^0)^T & b_0 \end{bmatrix}^T \quad (24)$$

$$\varphi_k(t) = \begin{bmatrix} (\varphi_k^0(t))^T & u_k(t) \end{bmatrix}^T \quad (25)$$

where $(\theta, \varphi_k(t)) \in R^{n+m+1}$. Define the tracking error at the k -th learning iteration as $e_k(t+1) = y_k(t+1) - r_{d,k}(t+1)$.

The adaptive learning control law follows from relations (4), (5) and (23)

$$P_{k-1}^{-1}(t) = P_{k-2}^{-1}(t) - F_{k-2}(t) + \alpha_{k-1}(t) \varphi_{k-1}(t) \varphi_{k-1}^T(t) \quad (26)$$

$$\hat{\theta}_k(t) = \hat{\theta}_{k-1}(t) + \alpha_{k-1}(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}(t+1) \quad (27)$$

$$u_k(t) = \frac{1}{\hat{b}_{0,k}(t)} (r_{d,k}(t+1) - (\varphi_k^0(t))^T \hat{\theta}_k^0(t)) \quad (28)$$

where t is discrete-time, k is the number of learning iteration. From relation (28), according to relations (24) and (25), it follows that

$$r_{d,k}(t+1) = (\varphi_k^0(t))^T \hat{\theta}_k^0(t) + \hat{b}_{0,k}(t) u_k(t) =$$

$$= \begin{bmatrix} (\varphi_k^0(t))^T & u_k(t) \end{bmatrix} \begin{bmatrix} \hat{\theta}_k^0(t) \\ \hat{b}_{0,k}(t) \end{bmatrix} = \varphi_k^T(t) \hat{\theta}_k \quad (29)$$

Using the plant model $y_k(t+1) = \varphi_k^T(t) \theta$, the tracking error can then be written as

$$e_k(t+1) = y_k(t+1) - r_{d,k}(t+1) =$$

$$= \varphi_k^T(t) \theta - \varphi_k^T(t) \hat{\theta}_k(t) =$$

$$= \varphi_k^T(t) (\theta - \hat{\theta}_k(t)) = \varphi_k^T(t) \tilde{\theta}_k(t) \quad (30)$$

with $\tilde{\theta}_k(t) = \theta - \hat{\theta}_k(t)$. From Lemma 2, relation (26) can be written as

$$P_{k-1}(t) = U_{k-2}(t) - \frac{U_{k-2}(t) \varphi_{k-1}(t) \varphi_{k-1}^T(t) U_{k-2}(t)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)} \quad (31)$$

$$U_{k-2}(t) = (P_{k-2}^{-1}(t) - F_{k-2}(t))^{-1} \quad (32)$$

Remark 3. In relation (28), the control gain estimate $\hat{b}_{0,k}$ must be nonzero for $k = 1, 2, 3, \dots$. We assume knowledge of the sign of b_0 (i.e. $b_0 > 0$) and upper bound on its magnitude (b_0). We introduce the constraint:

$$\hat{b}_{0,k} = \begin{cases} \hat{b}_{0,k}, & \hat{b}_{0,k} \geq b_{\min} \\ b_{\min}, & \hat{b}_{0,k} < b_{\min} \end{cases} \quad (33)$$

where $b_{\min}$ is known. In the applications, $\hat{b}_{0,k}$ is usually selected based on this constraint. Reference [33] proves that a priori knowledge of the sign of the system's high-frequency gain and an upper bound on its magnitude is not necessary for global system stability.

Remark 4. The adaptive control strategy is valid for minimum-phase systems, meaning the zeros of the polynomial $B(z^{-1})$ lie inside the closed unit disk. A slight relaxation of the constraints can be achieved by using the weighted one-step-ahead criterion instead of (20) [23]:

$$J_{\theta,k}(t+1) = (y_k(t+1) - r_{d,k}(t+1))^2 + \lambda u_k^2(t) \quad (34)$$

In this case, the condition for stable adaptive ILC requires that instead of $B(z^{-1})$, the following polynomial is minimum phase:

$$B(z^{-1}) + \frac{1}{b_0} A(z^{-1}) \quad (35)$$

For nonminimum-phase systems (where the zeros of $B(z^{-1})$ can lie inside the unit circle), reference [34] proposes the use of lifted-time stable inversion (stable inversion).

4. CONVERGENCE ANALYSIS

In this part of the paper, we address the stability of the adaptive Iterative Learning Control (ILC) algorithm [32-34]. The parameter updating law (27) operates iteratively along the iteration axis and recursively along the time axis.

We have adapted the result from ([21]; Lemma 6.2.1, p.181) to our problem. This adaptation leads to the following Lemma 3:

Lemma 3. Let us $t \in [0, 1, \dots, T]$ represent the length of the reference trajectory, and $k = 1, 2, 3, \dots$ be the number of learning iterations. The vector of measurements $\varphi_k(t) \in R^{n+m+1}$ and the tracking error $e_k(t) \in R^1$ are defined. Assume that the following conditions are satisfied:

1) $\{\alpha_k(t)\}$ is a sequence of positive numbers,

$$2) \lim_{k \rightarrow \infty} \frac{e_k^2(t+1)}{\alpha_k^{-1}(t) + \lambda_{\max} \left\{ \left(P_{k-1}^{-1}(t) - F_{k-1}(t) \right)^{-1} \right\} \|\varphi_k(t)\|^2} = 0$$

where $F_{k-1}(t)$ is the forgetting matrix and $P_{k-1}(t)$ covariance matrix,

$$3) \|\varphi_k(t)\| \leq k_1 + k_2 \max_{i \in [0, t]} |e_k(i+1)|$$

$$k_1 \in (0, \infty), \quad k_2 \in (0, \infty).$$

Then, the following results hold:

$$i) \quad \lim_{k \rightarrow \infty} e_k(t) = 0$$

$$ii) \quad \|\varphi_k(t)\| \text{ is bounded.}$$

Now, we present the main result of the paper:

Theorem. Let the system (15) and adaptive ILC (26)-(28) be considered, where the reference trajectory has length $t \in [0, 1, \dots, T]$, and $k = 1, 2, 3, \dots$ denotes the iteration number. Assume that the following conditions are satisfied for the adaptive ILC:

1⁰ The upper bounds of the degrees of polynomials $A(q^{-1})$ and $B(q^{-1})$, denoted by n and m , are known,

2⁰ The polynomials $A(z^{-1})$ and $B(z^{-1})$ are stable,

3⁰ The sign of the system's gain and an upper bound on its magnitude are known (as given in relation (32)),

4⁰ The length of the reference trajectory is constant for all k (for $\forall k$),

5⁰ The reference trajectory $r_{d,k}(t)$ is uniformly bounded for each k ,

6⁰ Initial $\varphi_k(0)$ is bounded for each k , i.e.

$$\|\varphi_k(0)\| \leq k_3 \text{ with } k_3 \in (0, \infty).$$

Then, the following results hold:

A) The estimate of the unknown parameter $\hat{\theta}_k(t)$ is uniformly bounded:

$$\|\hat{\theta}_k(t)\| \leq \frac{\lambda_{\max}\{P_{k-1}^{-1}(t) - F_{k-1}(t)\}}{\lambda_{\min}\{P_{k-1}^{-1}(t) - F_{k-1}(t)\}} \|\tilde{\theta}_0(t)\|$$

where $\hat{\theta}_k(t) = \theta - \hat{\theta}_k(t)$, $F_k(\cdot)$ and $P_k(\cdot)$ are the forgetting matrix and covariance matrix, and $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are minimum and maximum eigenvalues of corresponding matrix, respectively.

B) The tracking error $e_k(t)$ satisfies:

$$\lim_{k \rightarrow \infty} e_k(t) = 0$$

C) The sequences $\{y_k(t)\}$ and $\{u_k(t)\}$ are uniformly bounded.

Proof: Let us introduce

$$\tilde{\theta}_k(t) = \theta - \hat{\theta}_k(t) \quad (36)$$

Additionally, introduce a nonnegative function:

$$V_k(t) = \tilde{\theta}_k^T(t) P_{k-1}^{-1}(t) \tilde{\theta}_k(t) \quad (37)$$

Using relations (26), (27), (30), (36), (37) we obtain:

$$V_k(t) =$$

$$\begin{aligned} & \left[\tilde{\theta}_{k-1}(t) - \alpha_{k-1}(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}(t+1) \right]^T \cdot \\ & P_{k-1}^{-1}(t) \left[\tilde{\theta}_{k-1}(t) - \alpha_{k-1}(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}(t+1) \right] = \\ & \left[\tilde{\theta}_{k-1}^T(t) P_{k-1}^{-1}(t) - \alpha_{k-1}(t) \varphi_{k-1}^T(t) e_{k-1}(t+1) \right] \cdot \\ & \left[\tilde{\theta}_{k-1}(t) - \alpha_{k-1}(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}(t+1) \right] = \\ & = \tilde{\theta}_{k-1}^T(t) P_{k-1}^{-1}(t) \tilde{\theta}_{k-1}(t) - \alpha_{k-1}(t) \varphi_{k-1}^T(t) \tilde{\theta}_{k-1}(t) e_{k-1}(t+1) - \\ & - \alpha_{k-1}(t) \tilde{\theta}_{k-1}^T(t) \varphi_{k-1}(t) e_{k-1}(t+1) \\ & + \alpha_{k-1}^2(t) e_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \\ & = \tilde{\theta}_{k-1}^T(t) P_{k-1}^{-1}(t) \tilde{\theta}_{k-1}(t) - \\ & - 2\alpha_{k-1}(t) \tilde{\theta}_{k-1}^T(t) \varphi_{k-1}(t) e_{k-1}(t+1) \\ & + \alpha_{k-1}^2(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \end{aligned}$$

Let us introduce the function

$$\begin{aligned} \Delta V_k(t) &= V_k(t) - V_{k-1}(t) \\ V_{k-1}(t) &= \tilde{\theta}_{k-1}^T(t) P_{k-2}^{-1}(t) \tilde{\theta}_{k-1}(t) \end{aligned} \quad (39)$$

From relations (38) and (39), we can get:

$$\begin{aligned} \Delta V_k(t) &= \tilde{\theta}_{k-1}^T(t) P_{k-1}^{-1}(t) \tilde{\theta}_{k-1}(t) - \\ & - 2\alpha_{k-1}(t) \tilde{\theta}_{k-1}^T(t) \varphi_{k-1}(t) e_{k-1}(t+1) + \\ & + \alpha_{k-1}^2(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \\ & - \tilde{\theta}_{k-1}^T(t) P_{k-2}^{-1}(t) \tilde{\theta}_{k-1}(t) = \\ & = \tilde{\theta}_{k-1}^T(t) \left(P_{k-1}^{-1}(t) - P_{k-2}^{-1}(t) \right) \tilde{\theta}_{k-1}(t) - \\ & - 2\alpha_{k-1}(t) \tilde{\theta}_{k-1}^T(t) \varphi_{k-1}(t) e_{k-1}(t+1) + \\ & + \alpha_{k-1}^2(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \\ & = \tilde{\theta}_{k-1}^T(t) \left(-F_{k-2}(t) + \alpha_{k-1}(t) \varphi_{k-1}(t) \varphi_{k-1}^T(t) \right) \tilde{\theta}_{k-1}(t) \\ & - 2\alpha_{k-1}(t) \tilde{\theta}_{k-1}^T(t) \varphi_{k-1}(t) e_{k-1}(t+1) + \\ & + \alpha_{k-1}^2(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \leq \\ & \alpha_{k-1}(t) \tilde{\theta}_{k-1}^T(t) \varphi_{k-1}(t) \varphi_{k-1}^T(t) \tilde{\theta}_{k-1}(t) \\ & - 2\alpha_{k-1}(t) \tilde{\theta}_{k-1}^T(t) \varphi_{k-1}(t) e_{k-1}(t+1) + \\ & + \alpha_{k-1}^2(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \end{aligned} \quad (40)$$

From relations (30) and (31), it follows that

$$\begin{aligned} \Delta V_k(t) &\leq \alpha_{k-1}(t) e_{k-1}^2(t+1) - 2\alpha_{k-1}(t) e_{k-1}^2(t+1) + \\ & + \alpha_{k-1}^2(t) \varphi_{k-1}^T(t+1) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \\ & = -\alpha_{k-1}(t) e_{k-1}^2(t+1) + \\ & + \alpha_{k-1}^2(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}^2(t+1) \\ & = -\alpha_{k-1}(t) \left(1 - \alpha_{k-1}(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) \right) e_{k-1}^2(t+1) \end{aligned} \quad (41)$$

Now we can get using Lemma 2 (42)

$$\begin{aligned}
& \alpha_{k-1}(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) \\
&= \alpha_{k-1}(t) \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t) \\
&- \alpha_{k-1}(t) \varphi_{k-1}^T(t) \frac{U_{k-2}(t) \varphi_{k-1}(t) \varphi_{k-1}^T(t) U_{k-2}(t)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)} \varphi_{k-1}(t) \\
&= \frac{\varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)}
\end{aligned}$$

From last relation it follows that (43)

$$\begin{aligned}
& 1 - \alpha_{k-1}(t) \varphi_{k-1}^T(t) P_{k-1}(t) \varphi_{k-1}(t) \\
&= 1 - \frac{\varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)} = \\
&= \frac{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t) - \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)} \\
&= \frac{\alpha_{k-1}^{-1}(t)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)}
\end{aligned}$$

Using relations (41) and (43) we have

$$\Delta V_k(t) = - \frac{e_{k-1}^2(t+1)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)} < 0 \quad (44)$$

From last relation it follows that $V_k(t)$ is a non-negative and nonincreasing function. From that observation we have

$$\begin{aligned}
& \tilde{\theta}_k^T(t) P_{k-1}^{-1}(t) \tilde{\theta}_k(t) \leq \tilde{\theta}_{k-1}^T(t) P_{k-2}^{-1}(t) \tilde{\theta}_{k-1}(t) \leq \dots \\
& \leq \tilde{\theta}_0^T(t) P_{-1}^{-1}(t) \tilde{\theta}_0(t)
\end{aligned} \quad (45)$$

Using relation (4) one can obtain

$$\begin{aligned}
& \lambda_{\min} \{P_k^{-1}(t)\} \geq \lambda_{\min} \{P_{k-1}^{-1}(t) - F_{k-1}(t)\} \geq \dots \\
& \geq \lambda_{\min} \{P_{-1}^{-1}(t) - F_{-1}(t)\}
\end{aligned} \quad (46)$$

where for a real symmetric matrix $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote its minimum and maximum eigenvalues, respectively. From relation (46) it follows

$$\begin{aligned}
& \lambda_{\min} \{P_{-1}^{-1}(t) - F_{-1}(t)\} \|\tilde{\theta}_k(t)\| \\
& \leq \lambda_{\min} \{P_{k-1}^{-1}(t) - F_{k-1}(t)\} \|\tilde{\theta}_k(t)\| \\
& \leq \tilde{\theta}_k^T(t) P_{k-1}^{-1}(t) \tilde{\theta}_k(t)
\end{aligned} \quad (47)$$

From relations (45) and (47), we obtain

$$\begin{aligned}
& \lambda_{\min} \{P_{-1}^{-1}(t) - F_{-1}(t)\} \|\tilde{\theta}_k(t)\|^2 \\
& \leq \tilde{\theta}_0^T(t) \{P_{-1}^{-1}(t) - F_{-1}(t)\} \tilde{\theta}_0^T(t) \\
& \leq \lambda_{\max} \{P_{-1}^{-1}(t) - F_{-1}(t)\} \|\tilde{\theta}_0(t)\|^2
\end{aligned} \quad (48)$$

From the last relation, we obtain

$$\|\tilde{\theta}_k(t)\|^2 \leq \frac{\lambda_{\max} \{P_{-1}^{-1}(t) - F_{-1}(t)\}}{\lambda_{\min} \{P_{-1}^{-1}(t) - F_{-1}(t)\}} \|\tilde{\theta}_0(t)\|^2 \quad (49)$$

This proves statement A) of the theorem.

We now derive a relation between the measurement vector and the tracking error. From relation (15) we have

$$u_k(t-1) = \frac{A(q^{-1})}{B(q^{-1})} y_k(t) \quad (50)$$

The system (50) is stable according with condition 2^o of theorem. From (50) it follows

$$u_k(j-1) \leq k_4 + k_5 \max_{i \in [0, t]} |y_k(i)|, \quad j \in [1, t] \quad (51)$$

From relations (19) and (51) we have (52)

$$\|\varphi_k^0(t)\| \leq k_6 \left(k_4 + (\max(1, k_5)) \max_{i \in [0, t]} |y_k(i)| \right), \quad k_6 = n + m$$

Using condition 3^o of theorem and relations (29), (33) and (49), it follows that

$$\begin{aligned}
& u_k(t) \leq b_{\min}^{-1} \left(|r_{d,k}(t+1)| + \|\varphi_k^0(t)\| \|\hat{\theta}_k^0(t)\| \right) \leq \\
& b_{\min}^{-1} \left(r_{\max} + k_7 \|\varphi_k^0(t)\| \right) = \\
& = b_{\min}^{-1} r_{\max} + b_{\min}^{-1} k_7 \|\varphi_k^0(t)\| = k_8 + k_9 \|\varphi_k^0(t)\|
\end{aligned} \quad (53)$$

where $|r_{d,k}(t)| \leq r_{\max}$, $r_{\max} \in (0, \infty)$, $\|\hat{\theta}_k^0(t)\| \leq k_7$,

$$k_8 = b_{\min}^{-1} r_{\max}, \quad k_9 = b_{\min}^{-1} k_7$$

Further we have

$$\|\varphi_k(t)\| \leq \|\varphi_k^0(t)\| + |u_k(t)| \quad (54)$$

Using relations (51) and (52) from relation (54) follows (55)

$$\begin{aligned}
& \|\varphi_k(t)\| \leq \|\varphi_k^0(t)\| + k_8 + k_9 \|\varphi_k^0(t)\| = \\
& = k_8 + (1 + k_9) \|\varphi_k^0(t)\| \leq k_8 + \\
& + (1 + k_9) k_6 \left(k_4 + (\max(1, k_5)) \max_{i \in [0, t]} |y_k(i)| \right) \\
& = k_8 + (1 + k_9) k_6 k_4 + (1 + k_9) k_6 \left((\max(1, k_5)) \max_{i \in [0, t]} |y_k(i)| \right) \\
& = k_{10} + k_{11} \max_{i \in [0, t]} |y_k(i)|
\end{aligned}$$

where

$$k_{10} = k_8 + (1 + k_9) k_6 k_4, \quad k_{11} = (1 + k_9) k_6 (\max(1, k_5))$$

From the definition of the tracking error, it follows that

$$|e_k(t)| \geq |y_k(t)| - |r_{k,d}(t)| \geq |y_k(t)| - r_{\max} \quad (56)$$

Using relations (55) and (56) we have

$$\begin{aligned}
& \|\varphi_k(t)\| \leq k_{10} + k_{11} \max_{i \in [0, t]} (|e_k(i)| + r_{\max}) = \\
& = k_{10} + k_{11} r_{\max} + k_{11} \max_{i \in [0, t]} (|e_k(i)|) = k_{12} + k_{11} \max_{i \in [0, t]} |e_k(i)|
\end{aligned} \quad (57)$$

From the last relation, it follows that

$$\begin{aligned} \|\varphi_k(t)\| &\leq k_{12} + k_{11} \left(|e_k(0)| + \max_{i \in [0, t]} |e_k(i+1)| \right) \\ &= k_{13} + k_{11} \max_{i \in [0, t]} |e_k(i+1)| \end{aligned} \quad (58)$$

From (44) one can get

$$\begin{aligned} V_k(t) - V_{k-1}(t) &= \\ &= -\frac{e_{k-1}^2(t+1)}{\alpha_{k-1}^{-1}(t) + \varphi_{k-1}^T(t) U_{k-2}(t) \varphi_{k-1}(t)} \leq 0. \end{aligned} \quad (59)$$

Summing both sides of the last inequality yields:

$$V_k(t) = V_0(t) - \sum_{j=1}^k \frac{e_{j-1}^2(t+1)}{\alpha_{j-1}^{-1}(t) + \varphi_{j-1}^T(t) U_{j-2}(t) \varphi_{j-1}(t)} \leq 0. \quad (60)$$

Let us notice that

$$V_0(t) = \tilde{\theta}_0^T(t) P_{k-1}^{-1}(t) \tilde{\theta}_0(t) \quad (61)$$

is bounded and $V_k(t)$ is nonnegative, according with relation (60). From these facts, it follows that

$$\lim_{k \rightarrow \infty} \sum_{j=1}^k \frac{e_{j-1}^2(t+1)}{\alpha_{j-1}^{-1}(t) + \varphi_{j-1}^T(t) U_{j-2}(t) \varphi_{j-1}(t)} \leq \infty \quad (62)$$

Then we have

$$\lim_{k \rightarrow \infty} \sum_{j=1}^k \frac{e_k(t+1)}{\left(\alpha_k^{-1}(t) + \varphi_k^T(t) U_{k-1}(t) \varphi_k(t) \right)^{1/2}} \leq 0, \quad \forall t \quad (63)$$

$t \in [0, 1, \dots, T-1]$

Note that

$$\begin{aligned} &\frac{e_k^2(t+1)}{\alpha_k^{-1}(t) + \varphi_k^T(t) U_{k-1}(t) \varphi_k(t)} \geq \\ &\geq \frac{e_k^2(t+1)}{\alpha_k^{-1}(t) + \lambda_{\max} \{U_{k-1}(t)\} \|\varphi_k(t)\|^2} \end{aligned} \quad (64)$$

where $U_{k-1}(t) = P_{k-1}^{-1}(t) - F_{k-1}(t)$. Using the last two relations we have

$$\begin{aligned} &\lim_{k \rightarrow \infty} \frac{e_k^2(t+1)}{\alpha_k^{-1}(t) + \varphi_k^T(t) U_{k-1}(t) \varphi_k(t)} = \\ &= \lim_{k \rightarrow \infty} \frac{|e_k(t+1)|}{\left(\alpha_k^{-1}(t) + \varphi_k^T(t) U_{k-1}(t) \varphi_k(t) \right)^{1/2}}. \\ &\lim_{k \rightarrow \infty} \frac{|e_k(t+1)|}{\left(\alpha_k^{-1}(t) + \varphi_k^T(t) U_{k-1}(t) \varphi_k(t) \right)^{1/2}} = 0 \end{aligned} \quad (65)$$

From (64) and (65) it follows that

$$\lim_{k \rightarrow \infty} \frac{e_k^2(t+1)}{\left(\alpha_k^{-1}(t) + \lambda_{\max} \{U_{k-1}(t)\} \|\varphi_k(t)\|^2 \right)} = 0 \quad (66)$$

Using relations (58) and (66) and Lemma 3, we can prove statements B) and C) of the theorem. Specifically, the tracking error $e_k(t)$ converges to zero (statement B of the theorem), as shown in (67):

$$\lim_{k \rightarrow \infty} e_k(t) = 0. \quad (67)$$

Additionally, output $\{y_k(t)\}$ and input $\{u_k(t)\}$ are bounded (statement C of the theorem).

Remark 5. The theorem proves the stability of adaptive ILC for systems with constant parameters.

The algorithm (26)-(28) includes a forgetting matrix, which is particularly relevant in this case [35,36], as its selection can affect the convergence rate of the estimated parameters.

Remark 6. The DARMA process with time-varying parameters has the form

$$A(q^{-1}, t) y_k(t) = q^{-1} B(q^{-1}, t) u_k(t)$$

where

$$\begin{aligned} A(q^{-1}, t) &= 1 + a_1(t) q^{-1} + \dots + a_n(t) q^{-n} \\ B(q^{-1}, t) &= b_0(t) + b_1(t) q^{-1} + \dots + b_m(t) q^{-m} \end{aligned}$$

and if the polynomials $A(q^{-1}, t)$ and $B(q^{-1}, t)$ are considered in condition 2° of the theorem, they are uniformly asymptotically stable. The theorem also holds for uniformly bounded time-varying parameters.

The forgetting matrix is particularly relevant for the identification of systems with time-varying parameters. However, the theorem presented above is established for DARMA systems with constant parameters. The time-varying example considered below is included only to illustrate the numerical behavior and tracking capability of the proposed algorithm beyond the proven setting. A rigorous extension to time-varying parameters would require additional assumptions on the parameter variation and a separate convergence analysis

Remark 7. The main task of adaptive controller is decision making in real time based on recursive identification (estimation (learning) of unknown system parameter). The main kinds of adaptive control are direct adaptive control [18] where controller parameter are updated directly from the tracking of adaptive error and indirect adaptive control [18] where it is first estimated the unknown system parameter on-line and then using those estimates to compute new controller setting. However, improved response of the adaptive system in terms of speed and accuracy, as well as robustness in the presence of perturbations, may be possible if combined direct and indirect adaptive strategies [37]. For the control of dynamical systems in the presence of large uncertainties is introduced multi model adaptive control [38].

5. SIMULATION STUDY

In this paper, the effectiveness of the proposed adaptive ILC algorithm is evaluated through simulations involving a time-varying reference trajectory and

varying initial conditions. In the considered case, the time-varying reference trajectory is defined as

$$r_{d,k}(t) = \frac{h_k}{5} \left[2 - 3 \sin^3 \left(\frac{2\pi t}{20} \right) \right] \sin \left(\frac{2\pi t}{20} \right)$$

where the values h_k , for $k = 1, 2, 3, \dots, N$, follow a uniform distribution over the interval $[-1, 1]$. The reference trajectories, along with their random variation across successive iterations, are illustrated in Fig. 1 and Fig. 2.

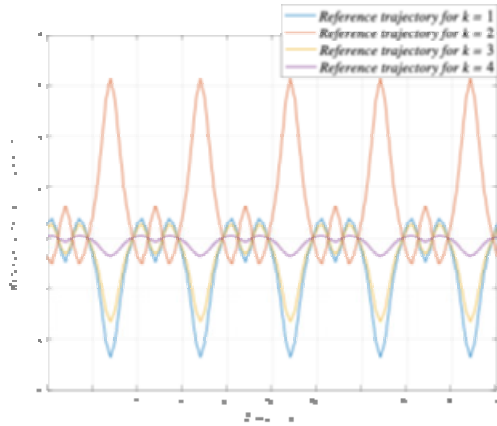


Fig. 1. Illustration of reference trajectory variability in successive iterations $k = 1, 2, 3, 4$

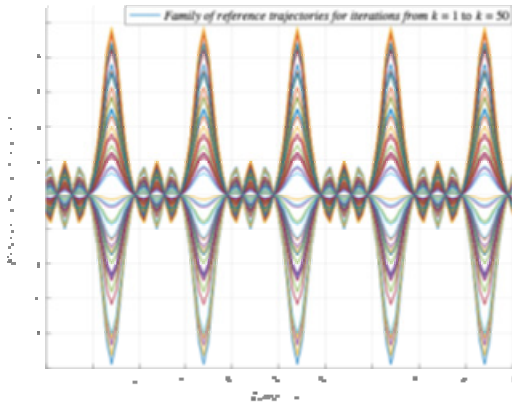


Fig. 2. Reference trajectory family used in 50 simulation iterations of system operation

The varying initial conditions are defined as random vectors $\varphi_k(0)$, for $k = 1, 2, 3, \dots, N$, whose elements are uniformly distributed over the interval $[-0.5, 0.5]$

$$\frac{1}{2} [\text{rand}(1, 200) - 0.5]$$

The norms of the vectors $\varphi_k(0)$ for $k = 1, 2, 3, \dots, 50$, are shown in the following figure.

In relation (26), we adopt

$$F_{k-2}(t) = (1 - \lambda) P_{k-2}^{-1}(t), \quad t = 0, 1, 2, \dots, T$$

$$\lambda \in (0, 1]$$

where λ is the forgetting factor. It is also assumed that the weighting factor is $\alpha(t) = 1$, for all $t = 0, 1, 2, \dots, T$. With these assumptions, the adaptation algorithm (26) – (28) takes the following form:

$$P_{k-1}^{-1}(t) = \lambda P_{k-2}^{-1}(t) + \varphi_{k-1}(t) \varphi_{k-1}^T(t)$$

$$\hat{\theta}_k(t) = \hat{\theta}_{k-1}(t) - P_{k-1}(t) \varphi_{k-1}(t) \cdot$$

$$\cdot (y_{k-1}(t+1) - \varphi_{k-1}^T(t) \hat{\theta}_{k-1}(t))$$

$$u_k(t) = \frac{1}{b_{0,k}} \left(r_{d,k}(t+1) - (\varphi_k^0(t))^T \hat{\theta}_k^0(t) \right)$$

The simulation analysis considers tracking errors, which should be minimized,

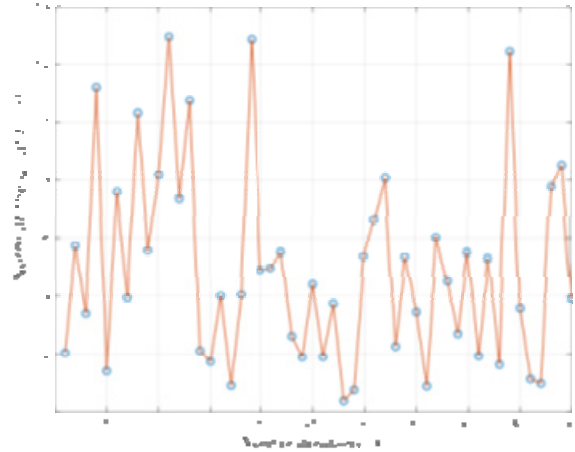


Fig. 3. Norm of the information vector φ at time $t = 0$ over iterations $k = 1$ to 50

$$e_k(t+1) = y_k(t+1) - r_{d,k}(t+1)$$

as well as the adaptive tracking errors,

$$e_k(t+1) = y_k(t+1) - \varphi_{k-1}^T(t) \hat{\theta}_{k-1}(t)$$

It is important to note that the errors $e_{k-1}(\cdot)$ and $\hat{e}_{k-1}(\cdot)$ are asymptotically identical. In all simulations, the following parameters were adopted:

$$b_{\min} = 0.1, \quad P_{-1}^{-1}(0) = 10^{-3} \cdot I, \quad \hat{\theta}_0(0) = 0$$

We consider two cases of controlled system: A) an LTI system (described by a DARMA model with constant parameters), and B) a system with time-varying parameters (described by a DARMA model with time-varying parameters).

A) DARMA model with constant parameters:

$$A(q^{-1}) = 1 - 1.6q^{-1} + 0.65q^{-2}$$

$$B(q^{-1}) = 0.4q^{-1} + 0.3q^{-2}$$

The DARMA model can be expressed by an iterative difference equation.

$$y_k(t+1) = 1.6y_k(t) - 0.65y_k(t-1) +$$

$$+ 0.3u_k(t-1) + 0.4u_k(t)$$

From the last relation, the following is obtained:

$$\varphi_k^0(t) = [-y_k(t), -y_k(t-1), u_k(t-1)]^T$$

$$\theta^0(t) = [-1.6, 0.65, 0.3]^T$$

And

$$\begin{aligned}\varphi_k(t) &= \left[\left(\varphi_k^0(t) \right)^T, u_k(t) \right]^T \\ &= \left[-y_k(t), -y_k(t-1), u_k(t-1), u_k(t) \right]^T \\ \theta &= \left[\left(\theta^0 \right)^T, b_0 \right]^T = [-1.6, 0.65, 0.3, 0.4]^T\end{aligned}$$

For the defined plant model, the tracking error is shown in the figure below.

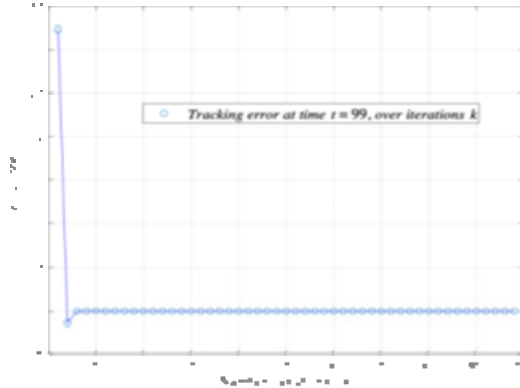


Fig. 4. Iteration-domain convergence of the tracking error at the representative time instant $t = 99$.

B) DARMA model with time-varying parameters. In this case, the iterative difference equation is:

$$\begin{aligned}y_k(t+1) &= a_1(t)y_k(t) + a_2(t)y_k(t-1) + \\ &+ b_1(t)u_k(t-1) + b_o(t)u_k(t)\end{aligned}$$

where the parameters are given by the following formulas:

$$\begin{aligned}a_1(t) &= -1.6 + 0.11 \sin\left(\frac{\pi t}{5}\right), \\ a_2(t) &= 0.65 + 0.065 \cos\left(\frac{\pi t}{10}\right) \\ b_o(t) &= 0.4, \quad b_1(t) = \begin{cases} 0.35, & t < 50 \\ 0.3, & t \geq 50 \end{cases}\end{aligned}$$

The graphical representation of the parameter variations is shown in the following figure.

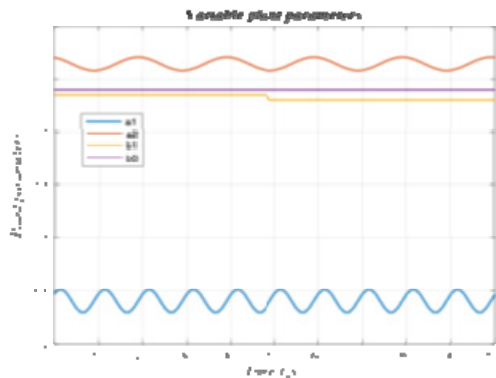


Fig 5. Time varying parameters of DARMA model

The adaptive tracking error $\hat{e}(t)$ for $\lambda = 0.99$ is shown in the following figure.

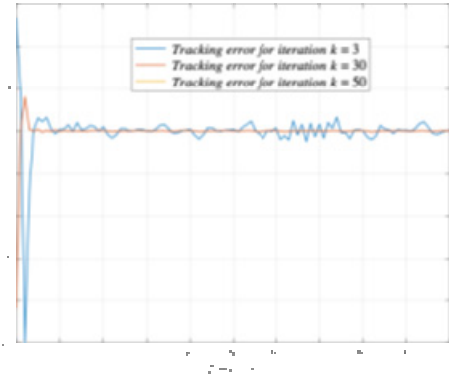


Fig. 6. Adaptive tracking error for $k = 3, 30, 50$; the curves for $k=30$ and $k=50$ are approximately identical

The simulation analysis of the plant model with varying parameters covers different forgetting factors λ . Fig. 7 shows the behavior of the adaptive tracking error $\hat{e}(t)$ for different forgetting factors ($\lambda = 1$ and $\lambda = 0.95$) in the first learning iteration ($k=1$).

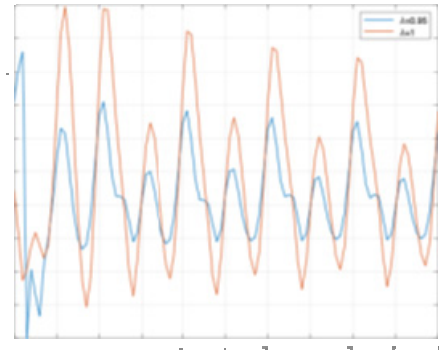


Fig. 7 Adaptive tracking error for different values of the forgetting factor ($\lambda = 1, \lambda = 0.95$) and $k=1$

6. CONCLUSIONS

This paper has presented an adaptive one-step-ahead Iterative Learning Control (ILC) methodology for discrete-time SISO systems described by DARMA models with unknown parameters. The proposed identification-control architecture integrates a generalized-forgetting recursive least-squares estimator with a one-step-ahead control law. In this way, parameter estimation is performed recursively along the time axis and iteratively along the learning axis, while the resulting parameter estimates are directly incorporated into the control law. The framework relaxes two conventional assumptions of ILC by allowing nonidentical initial conditions and reference trajectories that vary from iteration to iteration, while preserving a fixed trial length.

The principal theoretical results establish, under the stated assumptions, the uniform boundedness of the estimated parameters, convergence of the tracking error to zero as the number of learning iterations increases, and boundedness of the corresponding input and output sequences. These results are obtained for DARMA systems with constant parameters under assumptions that include bounded initial regressors and reference trajectories, a constant trial length, the required stabi-

lity/minimum-phase properties, and available information concerning the sign and a bound of the relevant control gain. These assumptions also indicate the principal practical restrictions of the proposed approach, particularly with respect to non-minimum-phase plants and systems for which the control direction is not known a priori.

An important feature of the proposed scheme is the generalized-forgetting mechanism in the recursive identification algorithm. In contrast to standard recursive least squares, forgetting prevents the covariance matrix from progressively losing its adaptation capability and therefore provides a mechanism for maintaining parameter adaptation when the information content of the data changes. This property is particularly relevant for systems whose parameters vary with time. However, the rigorous convergence theorem developed in this paper is established for systems with constant parameters. Accordingly, the numerical example with time-varying DARMA parameters should be interpreted as an illustration of the adaptation capability and practical potential of the generalized-forgetting scheme, rather than as a direct numerical verification of the theorem for the time-varying case. A rigorous extension of the convergence analysis to time-varying systems remains an important topic for future investigation.

Several further research directions follow naturally from the presented results. These include: (a) investigation of alternative control strategies within the adaptive ILC framework; (b) development of adaptive ILC schemes for non-minimum-phase systems; (c) removal of the requirement for a priori knowledge of the control direction and control-gain information; (d) extension of the methodology to multivariable and nonlinear systems; and (e) consideration of systems with iteration-varying trial lengths. Another promising direction is to investigate whether the recursive least-squares estimator can be replaced by computationally efficient accelerated first-order recursive learning algorithms, including momentum- and heavy-ball-type methods, while retaining suitable stability and convergence properties of the resulting adaptive ILC scheme.

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APPENDIX

Table 1. Adaptive ILC Regulator: Algorithmic Structure

<i>Algorithm for adaptive ILC</i>
Given/design parameters and initialization: $N, T, b_{\min} > 0, \varepsilon > 0, P_{-1}^{-1}(0) = \varepsilon \cdot I, \hat{\theta}_0(0)$ $\{\alpha_j(t) > 0\}, \{F_j(t) = F_j^T(t) \geq 0\}$ satisfying the conditions of Lemma 2, $\{r_{d,k,t}\}$ and the initial regressor $\varphi_k(0)$ (i.e. initial I/O data required to construct it) for each trial. An initial trial $k=0$ is assumed to be available. The true plant parameter vector θ is used only for simulation and is not an input to the adaptive controller.
<pre> for k = 1 : N for t = 0 : (T - 1) $e_{k-1}(t+1) = y_{k-1}(t+1) - r_{d,k-1}(t+1)$ $P_{k-1}^{-1}(t) = P_{k-2}^{-1}(t) - F_{k-2}(t) + \alpha_{k-1} \varphi_{k-1}(t) \varphi_{k-1}^T(t)$ $P_{k-2}^{-1}(t+1) = P_{k-1}^{-1}(t)$ $\hat{\theta}_k(t) = \hat{\theta}_{k-1}(t) + \alpha_{k-1}(t) P_{k-1}(t) \varphi_{k-1}(t) e_{k-1}(t+1)$ $\hat{\theta}_{k-1}(t+1) = \hat{\theta}_k(t)$ $\hat{b}_{o,k} = \begin{cases} \hat{b}_{o,k}, & \hat{b}_{o,k} \geq b_{\min} \\ b_{\min}, & \hat{b}_{o,k} < b_{\min} \end{cases}$ $u_k(t) = \frac{1}{\hat{b}_{0,k}} \left(r_{d,k}(t+1) - \left(\varphi_k^0(t) \right)^T \hat{\theta}_k^0(t) \right)$ Apply $u_k(t)$, measure $y_k(t+1)$, and update $\varphi_k(t+1)$ end $P_{k-1}^{-1}(0) = P_{k-1}^{-1}(T-1)$ $\hat{\theta}_k(0) = \hat{\theta}(T-1)$ end </pre>

**АДАПТИВНО ИТЕРАТИВНО УПРАВЉАЊЕ
УЧЕЊЕМ ЈЕДАН КОРАК УНАПРЕД ЗАСНО-
ВАНО НА РЕКУРЗИВНОЈ МЕТОДИ НАЈМА-
ЊИХ КВАДРАТА СА ГЕНЕРАЛИЗОВАНИМ
ЗАБОРАВЉАЊЕМ**

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Овај рад разматра адаптивно итеративно управљање учењем (ILC) један корак унапред за системе описане детерминистичким ауторегресивним моделом са покретним просеком (DARMA). Непознати параметри модела процењују се коришћењем недавно уведене методе рекурзивних најмањих квад-

рата са генерализованим заборављањем. Претпоставља се да је систем минималне фазе. Рад ублажава две основне претпоставке ILC-а: идентични почетни услови и идентичне референтне путање. Уместо тога, почетни вектор регресора се сматра случајним, а референтна путања варира током итерација. Предложени алгоритам има 2Д структуру, где се ажурирања параметара врше итеративно дуж осе итерације и рекурзивно дуж временске осе.